Derivatives, Prediction and True Fat Tails (i.e. Fractal), Part 1: The Fragility of Option Pricing

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This is a working notebook -- it cannot be quoted in this present version.

Derivatives that depends on the high consequential large deviation are marred with huge sampling error. I examine the sensitivity of the derivatives to the parameters, the sampling error of the estimations or "predictions", then look at the empirical stability of these parameters.

Organization: First 1) I do the math of distribution & derivatives, as there is no intelligent literature on the subject outside of the inapplicable Levy-Stable, 2) I show the magnitudes w.r. to some parameters (mainly the tail exponent \( \alpha \)) 3) I discuss the error in the estimation of these parameters.

Main point: For options on remote events, a small change in the tail exponent say \( \alpha \) between 1.5 and 2, well within the estimation errors, make the option change in value: a .5 change in exponent makes the error on the event vary by a factor >10, often >100. Moral: don't play with tail estimations, and don't believe that options can estimate anything.

1- True Fat Tails and Derivatives Pricing

Definition: true fat tails (see lecture x) are as follows \( P_{\alpha X}/P_X \) depends on \( n \), not \( X \) for \( X \) large enough.

First, we select a distribution without a tail-characteristic scale for \( x \) on the real line \(-\infty \) and \( \infty \), which consists in a fractal tails with exponent \( \alpha \) and a multiplying scale. Typical Student \( T \)

\[
\phi (u) = \frac{1}{\sqrt{\alpha \beta (\frac{\alpha}{2}, \frac{1}{2})}} \left( \frac{\alpha}{\alpha + u^2} \right)^{\frac{\alpha}{2}}, \quad u \in [-\infty, \infty], \quad \alpha \geq 1
\]

So for large \( u \) "in the tails", we can see that it behaves \( K u^{-\alpha-1} \), where \( K \) is a constant.

Where \( \beta (.) \) is Euler beta function \( \beta (a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt \).

\( \phi(\cdot) \) has fractal tails with exponent \( \alpha \) on both sides.

Note: I ignore the designation "Levy-stable"

\( \alpha = 5/2; \) Comparison with a Gaussian \( N(0, \sqrt{5/2}) \)
Comparison with an equivalent Gaussian

Now consider the multiplicative (monoperiodic) process $X = X_0 (1 + s \ u - c)$, where $u$ is $\phi$ distributed. $\frac{X - X_0}{X_0} - \frac{1}{c}$ is a straight relative price change with a "drift" term $c$ and a "dispersion" constant $s$ to scale by the "volatility", simplified as a multiple of mean deviation (for a given period between an initial 0 and $T$.) The problem is that we cannot take a fractal tailed distribution for $\log[\frac{X}{X_0}]$ for obvious reasons (too unwieldy; I tried), so we have to be content with relative price changes.

By change of stochastic variables, I am able to get the distribution of $X$, conditional on $X_0$.

(If $x$ has distribution $f$ then $y = z(x)$ has density $\frac{f(y(x))}{f(z(x))}$ where $g$ is the inverse function of $z$).

$$f(X) = \frac{1}{\sqrt{\alpha \ s \ X_0 \ \beta(\alpha^2, \ \frac{\alpha}{2})}} \left( \frac{\alpha}{(x - c \ X_0 - X_0^2) + \alpha} \right)^{(\alpha + 1) \ \frac{1}{2}}, \quad X > 0$$

**Caveat 1 and Renormalization**: The distribution $f(X)$ may have minutely small mass for $X < 0$, when $(1 + s \ u)$ turns negative, $s \ u < 1$. This requires an atrocially huge volatility and can be compensated by a truncating effect and renormalization of the mass with $f'[x] = f(x) \frac{1}{1 - \int_{-\infty}^x f'(x) \ dx}$. I left it out as it does not affect the exercise.

Indeed

$$\int_{-\infty}^0 f(X) \left|_{\alpha=2} \right. dX = \frac{1}{2} - \frac{1}{2 \sqrt{1 + 2 \ s^2}}$$

and

$$\int_{-\infty}^0 f(X) \left|_{\alpha=3} \right. dX (6 \ s^2 + 2) \sin^{-1} \left( \frac{\sqrt{3 \ s^2 + 1} - 1}{\sqrt{2 \ \sqrt{3 \ s^2 + 1}}} \right) = \frac{\sqrt{3} \ s}{3 \ \pi \ s^2 + \pi}$$

Which is very small: of the order of $< .01\%$ for high volatility environments when $\alpha=3$, and $.05\%$ when $\alpha=2$ -- thus justifying ignoring the renormalization.
Caveat 2 and Explosive Mean: Likewise the mean may become explosive upwards, in which case the compensation can be part of the drift $c$ just like the lognormal is compensated by a negative $-\frac{1}{2} \sigma^2$ (where $\sigma$ is the Gaussian standard deviation. But, for the purposes of the exercise $f(.)$ works well in addressing option errors.

\[
f[X] \quad \text{Distribution for the price } X
\]

Scalable as far as meets the eye (unlike the Lognormal)

\[
\text{Log } P > x
\]

I was not able to find a close solution except for $\alpha=2,3$. At $\alpha=\infty$ we get the standard Bachelier-Thorp (a.k.a. Black-Scholes) equation.

Note on moments: With no drift $c$,

**With finite variance** $\alpha=3$

\[
\alpha = 3,
\]

\[
E[X] = X_0 \left( \frac{2 \sqrt{3} s + 2 \cot^{-1} \left( \sqrt{3} s \right) + \frac{\pi}{2}}{2 \pi} \right) = X_0
\]
where \( \cot^{-1}(z) \) is the Arc Cotangent of \( z \)

\[
\begin{align*}
E \left[ \left( \frac{X - X_0}{X_0} \right)^2 \right] &= \frac{3 \ s^2}{2} \\
E \left[ \left| \frac{X - X_0}{X_0} \right| \right] &= \frac{\sqrt{3} \ s^2 (2 + 3 \ s^2)}{\pi + 3 \ s^2} 
\end{align*}
\]

With infinite variance (borderline) \( \alpha = 2 \)

\[
\alpha = 2, \ E[X] = \frac{X_0(s + \sqrt{2s^4 + s^2})}{2s} \approx X_0 \\
E \left[ \left| \frac{X - X_0}{X_0} \right| \right] = s \left( \sqrt{2} - \frac{s}{\sqrt{1 + 2s^2}} \right)
\]

### Call Options Under Different Parametrizations

**a- Call Option Price C with a Cubic \( \alpha \)**

Call Price \( C = \int_K^{\infty} (X - K) f(X) \, dX \mid \alpha = 3 \)

\[
C_3 = \frac{s \ X_0 \left( \frac{\pi \sqrt{\frac{1}{s^2 X_0^2}}}{\sqrt[3]{2}} \right) \left\{ 2 \sqrt{2} \left( 5 \sqrt{s \ X_0} \sqrt{5 \ s^3 \ X_0^2 + 2 (-K + C \ X_0 + X_0)^2} + \frac{\sqrt{5} \ s \ X_0 (-K + C \ X_0 + X_0)^2 (\xi_1 + 5) \xi_2}{5 \ s^2 \ X_0^2 + 2 (-K + C \ X_0 + X_0)^2} \right) + \frac{5^{3/4} \ \sqrt{\pi} (-K + C \ X_0 + X_0) \Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} \} \Gamma\left(\frac{7}{4}\right) \}
\]

\[
\begin{align*}
\xi_1 &= \frac{2 (-K + C \ X_0 + X_0)^3}{s^2 \ X_0^2} \\
\xi_2 &= _2F_1 \left( \frac{1}{2}, \frac{3}{4}; \frac{3}{2}; \frac{2 (-K + C \ X_0 + X_0)^2}{5 \ s^2 \ X_0^2} \right)
\end{align*}
\]

where \( _2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \).

**b- Call Option Price with \( \alpha = 5/2 \)**

\[
C_{5/2} = \frac{1}{6 \ 5^{3/4} \ \sqrt{\pi} \ \Gamma(\frac{3}{4})} \left\{ \frac{1}{2} \left( 2 \sqrt{2} \left( 5 \sqrt{s \ X_0} \sqrt{5 \ s^3 \ X_0^2 + 2 (-K + C \ X_0 + X_0)^2} + \frac{\sqrt{5} \ s \ X_0 (-K + C \ X_0 + X_0)^2 (\xi_1 + 5) \xi_2}{5 \ s^2 \ X_0^2 + 2 (-K + C \ X_0 + X_0)^2} \right) + \frac{5^{3/4} \ \sqrt{\pi} (-K + C \ X_0 + X_0) \Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} \} \Gamma\left(\frac{7}{4}\right) \}
\]

I apologize for the inelegance but I can't do better.

**c- Call Option Price with square \( \alpha \)**
\[ C_2 = \frac{1}{2} (\zeta^3 K^2 + (-2 (C + 1) X0 \zeta^3 - 1) K + X0 (C + ((C + 1)^2 + 2 s^2) X0 \zeta^3 + 1)) \]

where

\[ \zeta^3 = \frac{1}{\sqrt{K^2 - 2 (1 + C) K X0 + ((1 + C)^2 + 2 s^2) X0^2}} \]

**Comparison to the Volatility Smile (Bachelier-Thorp, a.k.a. Black Scholes)**

**The Infinite Variance Case:** \( \alpha \approx 2 \) does not mean anything for option pricing, it generates a volatility surface -- so long as the scaling \( s \) is calibrated on the absolute first moment.

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**2- True Fat Tails and Derivatives Errors**

By changing \( \alpha \) and maintaining the rest constant, we can do guess the consequence of a small error in the tail exponent on the option value.
Note that almost all options converge to the same price (minus moniness) when alpha drops to close to 1.

**Errors in Alpha Estimation**

The Mean Deviation= .42 for an estimated alpha of 2.62 (using the Hill Estimator).