# Extreme Value Theory: Fuhgetaboudit

### A SHORT PEDAGOGICAL NOTE

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Extreme Value Theory has been considered a panacea for dealing with extreme events by a bunch of "risk modelers" who work from models to reality and thing they understand *The Black Swan* and have a solution for the problem. On paper it looks great. But only on paper. The problem is the calibration and parameter uncertainty --in the real world we don't know the parameters. The ranges in the probabilities generated we get are monstrous. This is a short presentation of the idea, followed by an exposition of the difficulty.

## What is Extreme Value Theory? A Simplified Exposition

### *Case 1, Thin Tailed Distribution*

Let us proceed with a simple example.

**The Extremum of a Gaussian variable**: Say we generate N Gaussian variables $\{Z_i\}_{i=1}^{N}$ with mean 0 and unitary standard deviation, and take the highest value we find. We take the upper bound $E_j$ for the N-size sample run j

$$E_j = \text{Max}\left\{Z_{i,j}\right\}_{i=1}^{N}$$

Assume we do so M times, to get M samples of maxima for the set E

$$E = \left\{\text{Max}\left\{Z_{i,j}\right\}_{i=1}^{N}\right\}_{j=1}^{M}$$

The next figure will plot a histogram of the result.

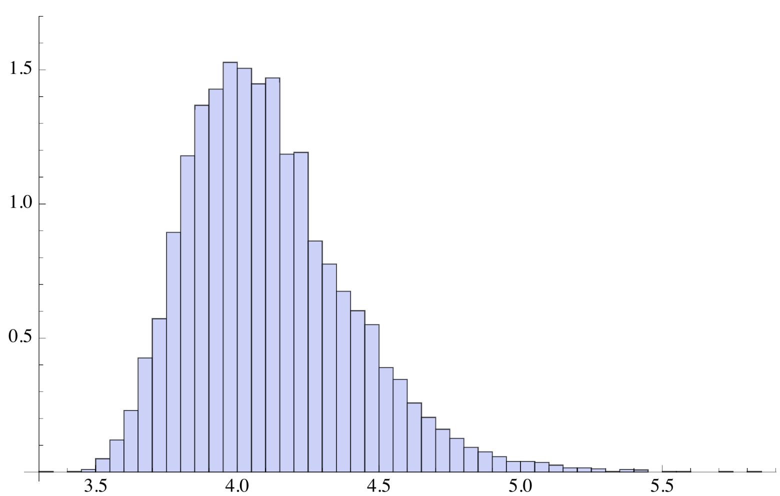


**Figure 1**: Taking M samples of Gaussian maxima; here N= 30,000, M=10,000. We get the Mean of the maxima = 4.11159 Standard Deviation= 0.286938; Median = 4.07344

Let us fit to the sample an Extreme Value Distribution (Gumbel) with location and scale parameters $\alpha$ and $\beta$, respectively: $f(x;\alpha,\beta) = \frac{e^{-e^{\frac{-x+\alpha}{\beta}}+\frac{-x+\alpha}{\beta}}}{\beta}$

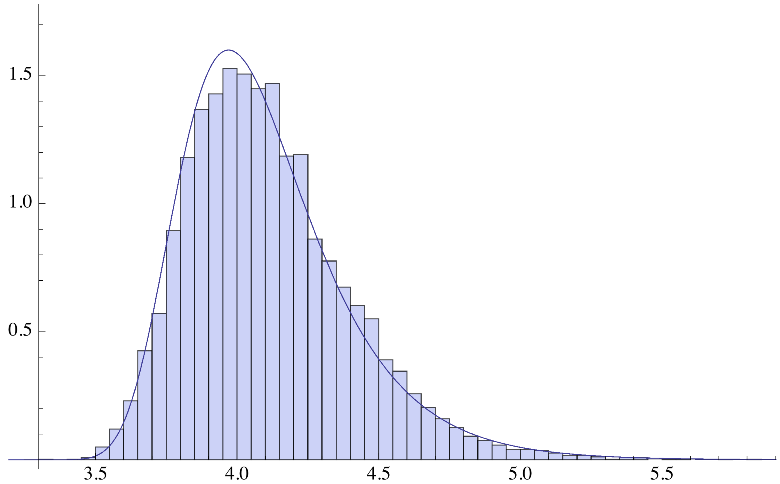


**Figure 2:** Fitting an extreme value distribution (Gumbel) $\alpha$= 3.97904, $\beta$= 0.235239

So far, beautiful. Let us next move to fat(ter) tails.

### *Case 2, Fat-Tailed Distribution*

Now let us generate, exactly as before, but change the distribution, with N random powerlaw distributed variables $z_i$, with tail exponent $\mu$=3, generated from a Student T Distribution with 3 degrees of freedom. Again, we take the upper bound. This time it is not the Gumbel, but the Fréchet distribution that would fit the result, Fréchet $\phi(x; \alpha,\beta) = \frac{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} \alpha \left(\frac{x}{\beta}\right)^{-1-\alpha}}{\beta}$, for x>0

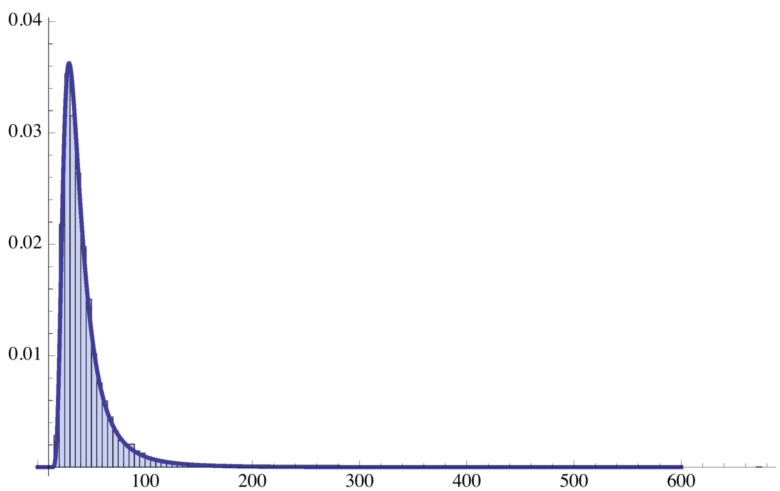


**Figure 3:** Fitting a Fréchet distribution tothe Student T generated with $\mu$=3 degrees of freedom. The Frechet distribution $\alpha$=3, $\beta$=32 fits up to higher values of E. But next two graphs shows the fit more closely.

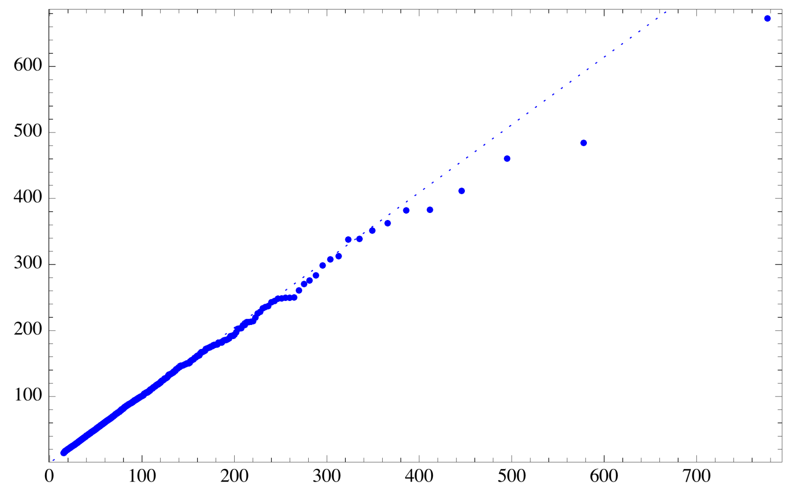


**Figure 5: Q-Q plot.** Fits up to extremely high values of E, the rest of course owing to sample insuficiency for extremely large values, a bias that typically causes the underestimation of tails, as the points tend to fall to the right.

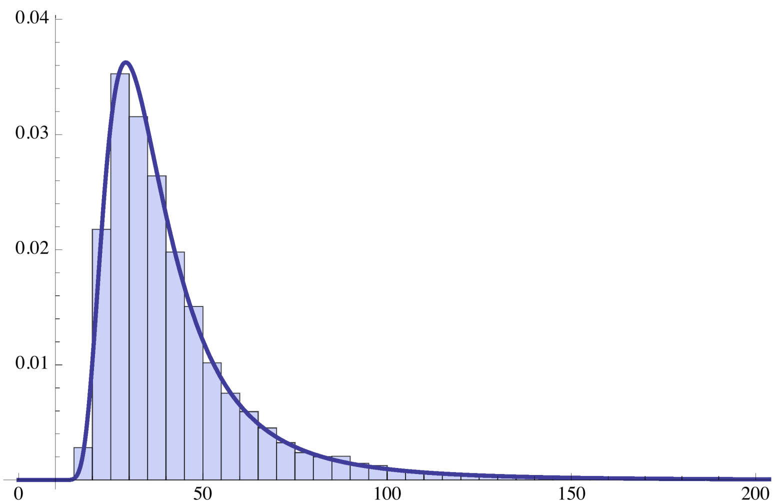


**Figure 4:** Seen more closely

# How Extreme Value Has a Severe Inverse Problem In the Real World

In the previous case we start with the distribution, with the assumed parameters, then get the corresponding values, as these "risk modelers" do. In the real world, we don't quite know the calibration, the $\alpha$ of the distribution, assuming (generously) that we know the distribution. So here we go with the inverse problem. The next table illustrates the different calibrations of $P_K$ the probabilities that the maximum exceeds a certain value K (as a multiple of $\beta$ under different values of K and $\alpha$.

| $\alpha$ | $\frac{1}{P_{>3\beta}}$ | $\frac{1}{P_{>10\beta}}$ | $\frac{1}{P_{>20\beta}}$ |
|---|---|---|---|
| 1. | 3.52773 | 10.5083 | 20.5042 |
| 1.25 | 4.46931 | 18.2875 | 42.7968 |
| 1.5 | 5.71218 | 32.1254 | 89.9437 |
| 1.75 | 7.3507 | 56.7356 | 189.649 |
| 2. | 9.50926 | 100.501 | 400.5 |
| 2.25 | 12.3517 | 178.328 | 846.397 |
| 2.5 | 16.0938 | 316.728 | 1789.35 |
| 2.75 | 21.0196 | 562.841 | 3783.47 |
| 3. | 27.5031 | 1000.5 | 8000.5 |
| 3.25 | 36.0363 | 1778.78 | 16 918.4 |
| 3.5 | 47.2672 | 3162.78 | 35 777.6 |
| 3.75 | 62.048 | 5623.91 | 75 659.8 |
| 4. | 81.501 | 10 000.5 | 160 000. |
| 4.25 | 107.103 | 17 783.3 | 338 359. |
| 4.5 | 140.797 | 31 623.3 | 715 542. |
| 4.75 | 185.141 | 56 234.6 | $1.51319 \times 10^6$ |
| 5. | 243.5 | 100 001. | $3.2 \times 10^6$ |

Consider that the error in estimating the $\alpha$ of a distribution is quite large, often > 1/2, and typically overestimated. **So we can see that we get the probabilities mixed up > an order of magnitude.** In other words the imprecision in the computation of the $\alpha$ compounds in the evaluation of the probabilities of extreme values.

I rest my case □