6. Dynamic Hedging and Volatility Expectation

Summary: The implied volatility derived from inverting the Black-Scholes equation to solve for the price of an option is not an unconditional forecast of future volatility (unless volatility is deterministic). It is only a forecast of the square root of the average variance of a biased set of sample paths for the underlying security—those paths that will affect the dynamic hedging of the option. Most research papers testing the “rationality” of the volatility implied from option prices naively miss the point. We compute the error to be large enough to invalidate a large number of empirical tests in the literature. An unconditional variance contract is created that is an unbiased rational predictor of future variance, and the properties of an accurate replicating portfolio are shown.

6.1. Introduction

An option trader meets a finance student at a street corner. “What is your expectation of volatility over the next year?” asked the future academic. “20 per

The author thanks Bruno Dupire for lengthy comments, as well as Samuel Wisnia for helpful discussions concerning the derivations below.
cent”, replied the trader. “At what BSIV (Black-Scholes implied volatility) would you sell a one-year European at-the-money straddle?” asked again the student.

“19.3 per cent”, answered the trader. Is the trader serious? Is his reply incoherent? Not at all. We will shed light on the question in this chapter. We will also show that empirical research on the efficiency of the volatility expectations using at-the-money options, by missing such a bias, present erroneous results. Without information about the entire spectrum of options, the “smile”, one cannot obtain information concerning the expectation of the volatility between two periods derived from option prices.

Beckers (1981), Day and Lewis (1993), Lamoureux and Lastrapes (1993), and Jorion (1995), present empirical tests involving the at-the-money option as naively taken as representative of some belief in unconditional volatility. Their research is grounded on the erroneous assumption that, if unbiased, the implied volatility derived from the price of the option would represents the traders’ consensus estimate of the future volatility. The representative sentence in Day and Lewis (1993) describes the method:\footnote{Jorion(1995) somewhat is aware that his testing may correspond to a misspecification of the Black-Scholes formula; yet he makes the same inference about expectations, unaware of the fact that one does not have to specify the model to understand that in an uncertain variance world, volatility expectations are conditional.}

“The comparison of the relative accuracy of the alternative forecasts of Futures market volatility indicate that the forecasts based on the implied volatilities...
volatility from the option market provide better forecasts of future volatility than do either historic volatility of the GARCH forecast of volatility” (emphasis mine).

Canina and Figlewski (1993) had the intuition of the bias when they conducted their study of the predictive power of listed options using a collection of options as a proxy of implied volatility, a collection that includes both at-the-money and away-from-the-money contracts. However such a set would clearly have the opposite effect of biasing the estimator upwards, since their out-of-the-money options are not weighted. Out-of-the-money options will trade higher than the exact estimator that we will show further down.

We conjecture that these methods are misleading; we show that it is impossible to derive any volatility expectation from a single option contract. In the presence of strong uncertainty concerning the distribution of volatility, measuring volatility requires stochastic volatility methods. These approaches would have been sound had they attempted to test beliefs in a deterministic volatility world; but then there would be a contradiction between a deterministic volatility environment and the reason for testing a volatility expectation. This point will be further discussed in chapter 7.

72 They looked at 8 strikes, individually, in a test of the predictive powers of volatility. It is worthy noting that neither options came close to being an acceptable predictor.
Chapter 1 formalized the activity of dynamic hedging in a pure Black-Scholes world, where the variance $V$ is both known and constant. Divergence from the $V$ in the valuation can only be caused by the tracking issue (see general appendix) as it is related to the frequency of dynamic hedging. We showed that the limit of the expectation of the dynamic hedging sequence (i.e. with an infinite number of revisions at infinitesimal time increments) leads to a deterministic portfolio – which allows for risk-neutral pricing. Under assumptions of dynamic completeness, the $V$ is a product freely available in the market through linear construction (a sum of strategies) and the operator can “lock-in” such a variance in a frictionless market.

The sections in the rest of this chapter are organized as follows.

2. We introduce the notion of a variance contract as a benchmark for a replicating portfolio.

3. We present some properties of the conditional distribution of asset prices under an unspecified stochastic volatility, using a one period $n$ states model.

4. We explore the properties of the replicating portfolio for the variance contract.

5. We study the parametric distribution of the variance as presented by the Hull and White.

Note that the general appendix discusses the economics of the volatility smile and the links to a state-dependent utility function – with a presentation of
Lucas (1978) asset pricing model. It will also include a critical review of the various analysis of the volatility “smile”, including a derivation of the results of Breeden and Litzenberger (1978), Dupire (1993), Derman and Kani (1993), and Rubinstein (1994).

6.2. A Variance Contract

What if a variance contract were traded in a market in the form of a forward? It would pay or receive the difference between the contract price and the realized volatility in the market between two dates (at some set sampling scale). We will next examine the properties of such a contract, particularly its relation to the expectation the future variance. If it is established that the contract needs to trade at the expectation of future variance, then an option that cannot completely replicate the payoff of the contract will not be the unbiased expectation of such a variance.

6.2.1. A Constant Volatility World

Assume the following price dynamics for asset $S$ with constant and time-independent volatility $\sigma$ and risk-neutral drift $m$ (use no discounting interest rates without loss of generality):

\[
\frac{dS_t}{S_t} = m dt + \sigma dZ^1
\]

Define $C(K,T,IV)$ as the Black-and Scholes (1973) solution for a European option on a asset $S$ for expiration period $T$, from time $t_0 = 0$. 
We remark that the Black Scholes implied volatility corresponds to the expectation of the volatility between $t_0$ and $T$. It may appear to be a tautology, except that even when volatility is deterministic, the average realized variance over a period of time will still exhibit a moderate deviation – according to the expected fourth moment of the distribution of the asset returns. Thus the variance in expectation of the realized volatility during the life of the option can be assumed to be a mere sampling issue, with convergence to a known volatility.

Define a $\Delta t$ period of observation, $T$ the expiration date of the option and $n$ the integer number of revision periods such that:

\[
(6-2) \quad n = \frac{T - t_0}{\Delta t}
\]

Take:

\[
(6-3) \quad V(\Delta t) = \frac{1}{n-1} \sum_{i=0}^{n} \left( \frac{S_{t_0 + (i+1) \Delta t}}{S_{t_0 + i \Delta t}} \right)^2 \left( \log \left( \frac{S_{t_0 + (i+1) \Delta t}}{S_{t_0 + i \Delta t}} \right) - \bar{R} \right)^2
\]

where $\bar{R}$ is the mean log return over the period.

**Definition 6.2-1** A forward variance contract $V(\Delta t)$ between two dates $t_0$ and $T$ pays or receives the difference between the realized variance measured at intervals $\Delta t$ and an initial value $V_0$.

This definition leads to the following expectation equation

\[
(6-4) \quad V_0 = E[V(\Delta t)|I_0] + \gamma
\]
The equation (6-4 resembles a conventional rational expectation model where $\gamma$ is the bias and $I_0$ the information set at time $I_0$. $\gamma$ is the premium for risk between period $t_0$ and $T$.

Next we consider the case of the Black-Scholes economy with known variance $V^*$, the equivalent of the perfect foresight in the rational expectations literature.

**Proposition 6-1 (scaling)** Under the assumption of homoskedasticity (constant $\sigma$), a measure of the difference between the sample $V_0$ will need to trade at the perfect foresight future variance $V^*$.

In addition we have the following properties:

*With $n$, the number of revisions, a positive integer,*

\[ a) \ E(V(\Delta t) - V^*) = 0 \text{ for all } \Delta t \]

\[ b) \ \lim_{\Delta t \to 0} \text{Var}(V(\Delta t) - V^*) = 0 \]

The property, discussed in Chapter 1, that an option dynamic hedging sequence is a *linear* combination of revisions should permit arbitrage pricing. Under the assumption that volatility is known and constant and given the frictionless environment, the fundamental theorem of asset pricing indicates that the arbitrage should drive the profit out of the trade, to compress the $\gamma$ to 0, whether $\gamma$ here is being defined as an “expectation bias” or a “risk premium”.
Clearly the fact that the Brownian motion is both homoskedastic, of known variance, and with independent increments\(^73\), makes the expectation of the variance independent of the scaling \(\Delta t\). This result is quite critical to show why we can still obtain the Black-Scholes price in a discrete time economy. It is a truism to say that a \(\Delta t\)-based policy of portfolio revision becomes a sampling issue (see Boyle and Emanuel, 1980).

We note two useful properties for the Black-Scholes equation:

**Property 1:** We can recover \(V^*\), the true variance (in expectation) regardless of the hedging policy \(\Delta t\).

**Property 2:** \(V^*\) should be reflected in every option, regardless of its strike price.

While \(\text{Var}(V^*)\) is a function of \(\Delta t\), it decreases at the speed \((1/n)^{1/2}\) (by a well known argument \(\text{Var}(V^*) = E(m_4^{1/2})\), where \(m_4\) is the fourth moment of the distribution). Using property 2 and the sampling invariance we get the result

\(^{73}\) The microstructure of markets is such that, alas, they are hardly of independent increments at a high frequency scale. We discuss in the general appendix the issue of bid-ask bounce and its effect on the first order negative autocorrelation. The implication of such negative autocorrelation is that when the variance measured at a short \(\Delta t\) scale is different from that measured at a longer \(\Delta t\) scale, some “market maker rent” become apparent. Typically, the microstructure of market is such that all the measurements of variance at a shorter frequency are biased upwards. Our empirical study of the variance in the C.B.T. U.S. bond futures, measured at 5 second intervals, show it to be as high as twice the close-to-close. The results are shown in the general appendix.
that, in the absence of any source of uncertainty attending a volatility contract, its value can be inferred from any option trading in the market.

To conclude, a rational expectation of volatility can be conducted when volatility is deterministic since if the continuous time dynamic hedging operator buys an option priced at an implied standard deviation lower than the Black-Scholes formula, he will stand to make a profit with probability 1. If he sells the options above such an IV, he will make a loss with probability 1. The expected profit of loss (from time $t_0$) is expected to be the derivative of the Black-Scholes option with respect to the variance times the difference between the true variance and the one used in the Black-Scholes formula. If the operator sells an option struck at $K$, at time $t_0$ using a Black-Scholes equivalent price $C(S_0,K,V')$, he will be expected to earn

$$\frac{\partial C}{\partial \nu}(V' - V^*) + \frac{1}{2} \frac{\partial^2 C}{\partial V^2}(V' - V^*)^2$$

### 6.2.2. Volatility is Not Constant

Now what if volatility were not constant? Using the result that the costs of replicating an option expiring at time $T$ depend on the average variance between $t_0$ and $T$. Property 1 would still hold if the average volatility along a sample path did not depend on the scaling. Property 2, however, would no longer apply.
We will see that it is no longer possible to replicate $V_0$ with an option. It would be possible, however, to assume there exists a variance contract in the market (these contracts indeed exist). Then, even under stochastic volatility, an option becomes redundant if it can be traded as a function of such a variance contract.

6.3. Presentation of the Regime Switching Representation

We start by relaxing the assumption of constant variance with the one-period regime switching model for the average variance. The regime switch intuition allows us to characterize stochastic variance without specification about the distribution of the underlying process for the variance itself. Naturally the well known stochastic volatility models as Hull and White (1987) become the continuous case of the one-period regime switching model with Gaussian densities. We will further expand the model in section {3.4} in order to discuss some more useful properties of a Markov switching process in place of a one period $T$ state price density.

Let us assume $R$ the $(n,1)$ vector of possible regimes for the average variance for the path of returns between $S_0$ and $S_T$, between time $t_0$ and $t$,

$$R^T = [r_1, r_2, \ldots r_n]$$

each regime having a unique associated variance. We denote $V$ the vector of variances $V_1$ through $V_n$, such that $V_i > V_j$ when $i < j$. Finally we impart to each regime $r_i$ a probability of occurrence $p_i$, with $P$ the corresponding vector and $p_i$ the
components. We assume that the components of vector $R$ are both exhaustive and mutually exclusive in their description of all the possible states. Thus

$$\sum_{i=1}^{n} p_i = 1$$

The expected average variance $\bar{V}$ will thus be $\sum_{i=1}^{n} p_i V_i$.

Assume normalization to no interest rates and drift to simplify. Therefore there are, between time $t_0$ and $T$, $n$ possible distinct distributions for the asset price:

$$g(S_T|R=r_i) = \frac{\log\left(\frac{S_T}{S_0}\right) + \left(\frac{1}{2} V_i \right) T}{\sqrt{V_i} \sqrt{T}}$$

with probability $p_i$

The unconditional distribution of the asset price is the marginal $g_s(S_T)$

$$g_s(S_T) = \sum_{i=1}^{n} g(S_T|R=r_i)p_i$$

From period $T$, and having seen the realization $S_T$, it is easy to infer the conditional probability of the market having been in any one of the regimes between the initial period and $T$, $P[R=r_i | S_T]$. We have

$$P[R=r_i | S_T] = \frac{p_i \cdot g(S_T|R=r_i)}{g_s(S_T)}$$

Therefore the expectation of the variance conditional on an ending price $S_T$
Dynamic Hedging and Volatility Expectation

\[
E[V | S_T ] = \frac{\sum_{i=1}^{n} V_i \cdot p_i \cdot g(S_T | R = r_i)}{g_s(S_T)}
\]

(6-9)

While the unconditional variance expectation is

\[
E[V] = \sum_{i=1}^{n} V_i \cdot p_i
\]

(6-10)

It can be easily shown that

\[
E[V | S_T = S_{ST}] < E[V]
\]

whenever the number of states \( n > 1 \). The intuition is that when the final sample path ends up very far away from \( S_0 \) the volatility expectation is highest – and conversely.

Using standard arguments about mixing distributions (see Feller, 1971) and owing to the fact that the realizations of \( R \) are independent from those of the asset price (though not the reverse since the magnitude of the move in the asset price will depend on the average volatility), one can use the pricing kernel \( g_s(S_T) \) as the term distribution, as the \( n \) densities for every state \( S_T \) are weighted with their probability of occurrence. A European call\(^{74}\) price becomes

\[
C = \int_{0}^{\infty} \text{Max}(S_T - K, 0) \left( \sum_{i=1}^{n} g(S_T | R = r_i) p_i \right) dS_T
\]

\(^{74}\) We use the notion European call generically, as all results can apply to the put using simple option algebra.
which is equivalent to

\[
\sum_{i=1}^{n} p_i \int_{0}^{\infty} \text{Max}(S_T - K, 0) \text{g}(S_T | R = r_i) dS_T
\]

hence the call value using the average variance becomes

\[
C(S_0, K, \bar{V}) = \sum_{i=1}^{n} p_i \text{BSC}(V_i)
\]

where BSC is the Black-Scholes valuation of a call using the variance \(V_i\). In other words, assuming only 2 regimes, an option price equals the option value if the regime equals regime 1 times the probability of being in regime 1 plus the option value if the regime equals regime 2 times the probability of being in regime 2.

**Example:** Take the following parameters:

\(n=2\)

\[
V = \begin{bmatrix} .06 \\ .02 \end{bmatrix}
\]

\[
P = \begin{bmatrix} .5 \\ .5 \end{bmatrix}
\]

\(t=1\)

The conditional probabilities of each of the two regimes \(r_1\) and \(r_2\) are shown in Figure 29 and Figure 30, conditional on the ending price \(S_T\) for the asset.
Figure 29 Conditional Probability of R = Regime 1. The horizontal axis shows the ending value $S_T$ and the vertical axis shows the corresponding conditional probability of being in the regime.

Figure 30 Conditional Probability of R = Regime 2. We note that the two figures are complementary and sum up to 1.

6.4. Properties of the Replicating Portfolio

The problem with an observer witnessing the price of options in the market is that they no longer reveal the expectation of $V$. What they reveal is far more subtle. An option with strike price $K$ will reveal the break-even volatility for the process assuming that the dynamic hedging was done at the expected variance weighted by
its sensitivity to the volatility of every sample path. We note that the partial
derivative of a Black-Scholes option with respect to the variance, given an initial
asset value \( S_0 \), corresponds to the limit of sensitivity of the costs of dynamically
hedging the portfolio to the variance each possible the sample path weighted by its
density.

Take the sample path starting at \( S_0 \) and expected to end at \( S_T \). Every final
state price has a unique expected variance associated with it. Using the subscript to
denote the derivative with respect to the variable,

\[
C_r(S, K) = \frac{C_{\sigma}(S, K)}{2\sigma}
\]

where the numerator is usually called the “vega”.

The equation (6-13) can be computed for a European option as

\[
C_v = \sum_{i=1}^{n} p_i \frac{S_0}{2\sqrt{V}} \sqrt{T - t_0} \, n[d1(V_i)]
\]

where \( n(.) \) is the normal density function and (again normalizing for the situation of
no drift and no interest rates) and \( d1(.) \) the familiar Black-Scholes shorthand:

\[
d1(V_i) = \log \frac{S_0}{V_i(T - t_0)} + \frac{1}{2}\sqrt{V_i(T - t_0)}
\]

Computing \( C_{vv} \) to find the maximum of the function yields (since \( C_v \) is concave)

\[
S^* = K \exp \left( -\frac{1}{2} \sum_{i=1}^{n} p_i V_i(T - t_0) \right)
\]
Since an option’s second derivative to variance is maximum at $S^*$, a higher impact on the option will come from the sample paths that end up around $K$ adjusted by the shift $\exp[\frac{1}{2} V T]$ attributed to lognormality. Figure 31 shows the plot of the derivative of the option with respect to variance at different $S_0$ levels.

*Figure 31 Sensitivity of the Option $C_v$ at the Initial Asset Levels $S_0$. We compare it to the flat sensitivity of the variance contract (since it concerns the returns not the price changes). The option’s sensitivity to variance is maximum near $S_0$.*

The option thus presents a $C_v$ (and consequently $C_{SS}$) that depends on $S_0$. This conditionality is not trivial as will see. Move next from $t_0$ to a period $t_s > t_0$ with associated asset price $S_s$. The sensitivity of the option will depend on the price $S_s$ while that of the remaining part of the variance contract will not be affected. An option trader who is long a given call struck at $K$ and short the variance contract will end up with an uncertainty in his portfolio. The net exposure can be seen on
Figure 31 by looking at the difference between the curve of $C_v$ and that of $V$. The figure shows that the hedge is only possible near the maximum $S^*$. 

*Proposition 6-2*

*A* - A variance contract cannot be replicated by a single option $C(S,K)$ when $p<1$ or $n>1$. 

*B* - There exists a linear combination of options that can replicate the variance contract. 

The proof resides in the fact that the rebalanced portfolio at scale $\Delta t$ no longer converges to $V^*$ for all sample path. $V$ the volatility contract is unconditional on $S_T$. The sensitivity to unconditional volatility $V$ for an option, $C_V$, is itself conditional upon the sample path

$$C_{V,S} = \exp \left( \frac{1}{2} \frac{(TV + \log(S/K))^2}{2TV} \left( TV - 2 \log\left( \frac{S}{K} \right) \right) \right) / 4\sqrt{2\pi} \sqrt{TV^{3/2}}$$

We further need to normalize the measure by $1/S$ in order to insure that the sensitivity of the option to the variance of the asset price remains constant.

Take a portfolio $\Pi$ constituted of a series of call option strikes, ranging between $K_L$ and $K_H$, with $\Delta K = \frac{K_H - K_L}{n}$
(6-17) \[ \Pi(S, K, K_L, K_H, n) = \sum_{i=1}^{n} \frac{C(K_L + i\Delta K) + P(K_L + i\Delta K)}{(K_L + \frac{i(K_H - K_L)}{n})^2} \]

Figure 32 Variance Sensitivity of 3 Portfolios at Different Initial Asset Price Levels. Parameters are \( V = 0.04 \) and \( T = 1 \) year. Portfolio \( \Pi_1 \) has \( K_L = 96, K_H = 104 \) and \( n = 10 \); Portfolio \( \Pi_2 \) has \( K_L = 80, K_H = 120 \) and \( n = 10 \); Portfolio \( \Pi_3 \) has \( K_L = 1, K_H = 219 \) and \( n = 100 \);

Figure 32 shows the variance sensitivity of three portfolios \( \Pi_v \), each of a different degrees of density between strikes. Clearly portfolio \( \Pi_3 \) has an even exposure to variance; it will therefore be the one that can be the candidate for the hedging of the unconditional variance. We will work with \( \Pi^* \), \( \Pi \)'s limiting value, as \( \Delta t \) goes to 0 and \( n \) tends to infinity, assuming continuous and infinite the strike prices exist. It is

(6-18) \[ \Pi^* = \int_0^\infty \frac{\Omega(K)}{K^2} dK \]

\(^{75}\) Note that the derivative of the package with respect to \( S = t N(-d_2(S, K_l, V)) - N(-d_2(S, K_H, V)) \), \( K_l \) is the lower bound and \( K_H \) the upper bound.
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To prove B in the proposition it suffices to show that $\Pi^*$ is insensitive to $S$, the final price. Then the portfolio $\Pi^*$ constituted of a continuum of strikes will be a perfect hedge against the unconditional variance, thus protecting the portfolio against all sample paths.

Next generate a “volatility smile”, i.e. the volatility for the equivalent Black-Scholes options as a function of $K$. $C(K, \sigma^2(K))$ is the new way we will write the call price. The markup over Black-Scholes per strike will be shown to be symmetric owing to the assumed independence between the states for volatility and those for the underlying asset. Owing to the convexity of the options that are away from the money, we have

$$\sigma^2(K_0 + \alpha) + \sigma^2(K_0 - \alpha) > \sigma^2(K_0)$$

where $\sigma^2(K)$ is the implied volatility of the strike price $K$ and $K_0$ the at-the-money option (such that $\log [F/K_0]=0$, where $F$ is the forward of $S$ here deemed equal to $S$ since we cancelled the interest rate effect).

**Remark:** $\sigma^2*$, the integral of the $\sigma^2(K)$ weighted by the derivative of the options struck at $K$ with respect to the variance will approximate the expectation of $V$.

(6-19)

$$\sigma^2* = \frac{1}{K} C_{1*}(K) \sigma^2(K) dK$$

We will show that $\int \frac{C(K, \sigma^2(K))}{K^2} dK$ can be approximated by $\int \frac{C(K, \sigma^2*)}{K^2} dK$.
6.5. Proofs and Approximations

6.5.1.1. Proof of the statement that the weighted strikes delivers $\sigma^2$ in a constant volatility world.

Take the Straddle

$$\Omega(K) \equiv P(K) + C(K)$$

We have a portfolio composed of a series of straddles

$$\Omega \equiv \int_{L}^{H} \frac{1}{K^2} \Omega(K) \, dK$$

with the portfolio long one unit of $\Omega$ against the hedge $\frac{\partial \Omega}{\partial S} S$

Assuming no interest rates in the economy without any loss of generality, and the following relation between the upper bound and the lower bound:

$$H = \frac{1}{L}$$

We have the expiration value

$$\Omega_T = \int_{L}^{S_T} \frac{1}{K^2}(S_T - K) \, dK + \int_{S_l}^{\frac{1}{L}} \frac{1}{K^2}(K - S_T) \, dK$$

since

$$\int_{L}^{S_T} \frac{1}{K^2}(S_T - K) \, dK = \frac{S_T}{L} + \log(L) - \log(S_T)$$
and
\[ \int_{S_T}^{1/L} \frac{1}{K^2} (K - S_T) \, dK = L S_T + \log \left( \frac{1}{L} \right) - \log(S_T) \]

\[ \frac{\partial \Omega_T}{S_T} = \frac{1}{L} + L - \frac{2}{S_T} \]

finally
\[ \Omega_T + \frac{\partial \Omega_T}{S_T} S_T = -2 \log(S_T) \]

\[ E \left( \Omega_T + \frac{\partial \Omega_T}{S_T} S_T \right) = \sigma^2 \Delta t \]

so long as \( S_T \) expires between \( L \) and \( 1/L \), which can be satisfied at the limit.

6.5.1.2. Approximation in a world where \( \sigma \) is a function of \( K \).

We need to find \( \sigma^* \) such that
\[ \int_{\mathbb{R}^+} \frac{1}{K^2} \Omega(S, K, \sigma(K)) \, dK = \int_{\mathbb{R}^+} \frac{1}{K^2} \Omega(S, K, \sigma^*) \, dK \]

\[ \int_{\mathbb{R}^+} \frac{1}{K^2} \Omega(S, K, \sigma(K)) \, dK = \int_{\mathbb{R}^+} \frac{1}{K^2} \left( \Omega(S, K, \sigma_0) + (\sigma^2(K) - \sigma_0^2) \frac{\partial \Omega(S, K, \sigma_0)}{\partial \sigma_0^2} \right) \, dK \]

where \( \sigma \) is the perfect forecast volatility,
\[ \frac{1}{2} \sigma^* \Delta t = \frac{1}{2} \sigma_0^2 \Delta t + \int \frac{1}{K^2} \sigma^2(K) \Omega_v(K) \, dK - \sigma_0^2 \int \frac{1}{K^2} \Omega_v(K) \, dK \]

since \( \int \frac{1}{K^2} \Omega_v(K) \, dK = \frac{1}{2} \Delta t \)

\[ \sigma^* = \int \frac{1}{K^2} \sigma^2(K) \Omega_v(K) \, dK \]

### 6.6. Example

We selected the following portfolio of options on the Chicago Mercantile Exchange, as represented by all the traded strikes. With the at-the-money option trading at a volatility of 20.4%, the unconditional volatility shows 21.5%, as reflected by the full “smile”.

*Figure 33 Variance Curve for the CME SP500 option on Futures on July 29, 1997, October expiration, 81 days. Future price 959. The at-the-money volatility is at 20.4%.*
Dynamic Hedging and Volatility Expectation

Table 2 Volatility Surface for SP500 on July 29, 1997.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Vol (%)</th>
<th>Var</th>
<th>Strike</th>
<th>Vol (%)</th>
<th>Var</th>
</tr>
</thead>
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<td>0.098</td>
<td>935</td>
<td>21.18</td>
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<td>20.65</td>
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<td>955</td>
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</tr>
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6.7. *Stochastic Volatility, Complete Markets and Knightian Uncertainty: a Brief Discussion*

The addition of uncertainty attending the $\sigma^2$ leads to a set of problems—not the least of which is that the existence of an additional source of uncertainty creates a pricing problem owing to the weakening of the risk neutrality argument\(^\text{76}\). The problem of an undiversifiable risk can be skirted intellectually, in some cases, as was done by Merton (1976), by assuming that for a Poisson process, ergodicity leads to the settling of the distribution to a known $\sigma^2$ for *every* sample path\(^\text{77}\). Such an argument is, alas, invalid with the Hull-White or purely stochastic volatility

\[^{76}\text{Among the works on non-constant volatility, the major contributions are by Merton (1973) who had a foreboding of the problem as the Ito process did not necessarily assume a constant volatility, further developed by Cox (1976) Constant Elasticity of Variance model, CEV. The effect of a double Brownian motion was investigated by Hull and White (1988), Scott (1992). Since Engle (1982) there has been an extremely rich literature on ARCH methods, with several investigations to option pricing theory. See the author's discussion of such methods in Taleb (1997).}\]

\[^{77}\text{Take a mixture of volatilities. The mixture will be expected at any point in time t to converge to }\sigma^2 = \sum W_i \sigma_i^2 \text{ (owing to the ergodicity of the Markov chain). See discussion in general appendix.}\]
models because volatility trajectories are of unit root and will not necessarily revert. In other words, using $E_t$ as the expectation operator at time $t$,

\[(6-20) \quad E_0(\sigma_T^2) = \sigma^2\]

in both cases, while we have

\[(6-21) \quad E_t(\sigma_T^2) = \sigma_t^2\]

in the Hull-White (and other stochastic volatility models), and

\[(6-22) \quad E_t(\sigma_T^2) = \sigma^2\]

in the Merton jump diffusion case. This point will be discussed in chapter 7. In the first case, the uncertainty is measurable, in the Knightian sense, while, in the second case, it is not.

It is appropriate here to dwell on Frank Knight’s well repeated differentiation between risk and uncertainty. Risk is what is measurable, can be calculated, when we know the probabilities. Risk cannot yield profits (a version of today’s market completeness). But uncertainty can, as it becomes the equivalent of an unknown the idea of which we know little.

The only “risk” that leads to profit is a unique uncertainty resulting from an exercise of ultimate responsibility which in its very nature cannot be insured nor capitalized nor salaried. (Knight, 1921)

Risk is measurable; uncertainty is not –this is where the distinction lies. A more subtle approach is to see that risk itself can be derived from the price paid for mutually exhaustive states of natures, thus creating an implied measure of risk in the same framework as the utility derivation. Thus an Arrow-Debreu price can turn
into a probability, thus making not measurable uncertainty turn into a measurable state price. It would suffice to have the price of a contingent claim paying a unit of currency given the occurrence of such state of nature in order to have the two concepts, measurable risks and non measurable uncertainty become one and the same. But what if we are dealing with absence of contingent payoffs? Clearly a market that that does not provide an Arrow-Debreu price is a market that has a distinction between risk and uncertainty.


We expand the previous asset price dynamics by taking a Hull-White(1987) stochastic volatility process where changes in asset price $S$ and its variance $V$ are the results of uncorrelated Brownian motions $Z^1$ and $Z^2$. The process can be written as

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dtZ^1
\]

\[
\frac{dV_T}{V_T} = \nu dZ^2
\]
Define the adjusted one-period equivalent volatility process between periods $t_0$ and $t$ as the stochastic integral

\[
V_T = \frac{1}{T} \int_0^T \sigma_s^2 \, ds
\]

The “volatility of volatility” $V_v$ corresponds to the volatility of the quadratic norm of the process for the local volatility. We recover the Black-Scholes process when $V_v = 0$.

We further write $t_0$ values $S_0$ and $\sigma_0$. Define $E$ as the expectation operator at time $t_0$.

Proposition 6-3 Assuming asset and volatility price dynamics (6-23 and (6-24, the expectation of the average variance of the process conditional on a terminal price $S_T$ has the following properties:

- a) $E(V_T | S_T = E(S_T)) < E(V_T)$ whenever $V_v > 0$

- b) $E(V_T | S_T = E(S_T)) - E(V_T | S_T = S_T^*) > 0$ whenever $|E(S_T) - S_T^*| > 0$

Rephrasing Proposition 6-3 a) means that whenever volatility is stochastic, with changes independent from the asset returns, volatility is expected to be lower when the asset price ends up at its expectation. Rephrasing Proposition 6-3 b) means that conditional volatility increases with the difference between terminal price $S_T$ and the expected terminal price.
The expectation of the average variance of the process conditional on a terminal price is

\begin{equation}
E(V_T | S_T) = \frac{\int_0^{\infty} f(S_T | V_T) V_T f(V_T) dV_T}{\int_0^{\infty} f(S_T | V_T) f(V_T) dV_T}
\end{equation}

The joint distribution of variance and asset price is

\begin{equation}
f(S_T, V_T) = f(S_T | V_T) f(V_T)
\end{equation}

since \( f(V_T) \) is the marginal distribution. Using (6-27), the distribution of \( V_T \) conditional on \( S_T \) becomes

\begin{equation}
f(V_T | S_T) = \frac{f(S_T | V_T) f(V_T)}{f_S(S_T)}
\end{equation}

where

\begin{equation}
f_S(S_T) = \int_0^{\infty} f(S_T | V_T) f(V_T) dV_T
\end{equation}

We note that \( f_S(S_T) \) is the pricing kernel; it is the equivalent to the Arrow-Debreu state-price density for period \( t \). It is today’s price for a security paying \( f_s(S_T) dS_T \) in the event of the state variable lying between \([S_T, S_T+dS_T]\) at period \( T \). Thus \( f_s(S_T) \) can be considered as a fat-tailed deterministic volatility process or a Gaussian kernel with stochastic volatility. We will see that for a one-period model (affecting path-independent contingent claims) both yield the same result.
From the process (under proper regularity conditions), thanks to the assumption of the independence between $Z^1$ and $Z^2$, we can infer the one-period state-price density for $S$ by standard methods as follows\(^{78}\):

$$f(S_T|V_T) = \frac{n\left(\frac{S_T}{S_0} + (\mu + \frac{1}{2} V_T)T\right)}{S_T \sqrt{V_T T}}$$

(6-30)

where $n(.)$ is the normal density function. The distribution of the variance at period $T$:

$$g(V_T) = \frac{n\left(\frac{V_T}{V_0} + \frac{1}{2} V_T^2 T\right)}{V_T \sqrt{V_T T}}$$

(6-31)

We will further interpret the last equation as the unconditional variance. We have $E(V_T) = V_0$ since it was assumed that there was no drift between $t_0$ and $T$.

Figure 34 plots the results of (6-26). There is indeed a dependence between $S_T$ and $V_T$ given that the level of $S_T$ depends on the volatility taken to get there. In other words, conditional on a large move, the most likely volatility to take $S_T$ is the highest one.

\(^{78}\) See lemma in Hull and White, 1987.
Dynamic Hedging and Volatility Expectation

Figure 34: Square Root of the Conditional expectation $\sqrt{\mathbb{E}[V_T|S_T]}$ using parameters $S_0=100$, $\sigma_0=.157$, $\nu=0.5$ $t=0.5$.

We note that the “fat tails” can be generated with the function $\Phi(S_T)$ which includes the expectation of a stochastic volatility. Figure 35 shows such fat tail state price density.

Figure 35: Pricing Kernels with $S_0=100$, $\sigma_0=.157$, $t=.5$ one with $V_v=0$, the other $V_v=.5$. The $V_v=.5$ causes the sharper peak and higher tails (different scales).
As to option pricing, we can obtain the value of a European option using (6-30) and (6-31). The European option becomes the unconditional integral across state prices. The price will be

\[ C(S_0, K, V_0, V_T, T) = \int_0^\infty \int_0^\infty \max(S_T - K, 0) f(S_T, V_T) dS_T dV_T \]

which is equivalent of the integral of a Black-Scholes option across volatility states

\[ C(S_0, K, V_0, V_T, T) = \int_0^\infty BSC(K, V_T) f(V_T) dV_T \]

Third, using the alternative Arrow-Debreu pricing kernel obtains

\[ C(S_0, K, V_0, V_T, T) = \int_0^\infty \max(S_T - K, 0) f(S_T) dS_T \]

We note the Breeden and Litzenberger (1978) result, where the second derivative of the European option with respect to the strike corresponds to the present value of the density of the state at expiration, an option value can yield some insight on the asset distribution (our inverse problem)

\[ f_S(S) \bigg|_{S=K} = \frac{\partial^2 C}{\partial K^2} \]

We can get the state price density at K, assuming that the inverted volatility \( \sigma \) varies with strike price: