2 Trading With a Stop

This lecture uses an option-theoretic approach to examine the properties of a dynamic trading strategy that includes a stop-loss rule.

2.1 INTRODUCTION

A trader holds a long position in the Japanese yen against the U.S. dollar. What is his risk? It is an unfortunate fact that most of the literature on risk management assumes that his risk is that of the yen. In reality, his strategy matters far more. If he trades with a stop loss, then his maximum risk becomes the difference between current market value and the level at which his stop loss was set, plus an additional variance called "slippage". Such slippage can be very significant--but it definitely escapes the methods of risk management in current use.

When the literature analyzes or discusses a market, the statistical properties of the latter are put forth--not those of the operator involved in trading the instrument. In other words, assume $S_t$ a security following a Brownian motion (the asset price dynamics are to be specified further down), with drift $\mu$ and variance $\sigma^2$, and $V_t$, a wealth process (that is, a stochastic integral). We know that under a non-anticipating dynamic strategy in which $\theta$ is the dollar proportion of wealth invested in the asset $S$, $V_t = \theta_0 \cdot S_0 + \int_0^t \theta_s dS_s$. However, the finance literature generally assumes that $V_t = \theta_0 \cdot S_t$, in which case the stochastic properties of $V_t$, namely the drift, variance and higher moments, will be indistinguishable from those of $S_t$ (multiplied by $\theta$). Nothing is further from reality: when $\theta_s$ is different from $\theta_0$, the properties of $V_t$ will depend on those of $\theta$ more than those of $S$.

More worrisome is the increased sophistication in researching the equivalent Value at Risk for the portfolio by the examination of the statistical properties of $S_t$, not those of $V_t$. Intuitively, a trader who holds a very risky security under a dynamic stop loss strategy would perhaps have a lower variance than a static
counterpart. In addition, the notion of variance may no longer hold as we are concerned with the higher moments.

2.1.1 Density Function for V

We largely use the results obtained from the literature on the pricing of derivative securities subjected to an absorbing barrier.

Consider an operator who has a position V invested at 100% in S the security. The dynamics of V will be indistinguishable from those of the returns of S. The density function \( p(z_t) \) of the returns of the sub-strategy between initial time \( t_0 = 0 \) and t will be decomposed in the following three parts:

1) The ordinary state price density function \( f(.) \)

Setting:

\[
(2-1) \quad z_t = \ln \left( \frac{S_t}{S_0} \right)
\]

\[
(2-2) \quad f(z_t, t) = \frac{n \left( \frac{z_t - \mu t}{\sigma \sqrt{t}} \right)}{\sigma \sqrt{t}}
\]

where \( n(.) \) is the normal density function, \( \mu \) the expected return from holding the portfolio*, \( \sigma^2 \) the variance (held constant during the period), \( t \) the time until expiration.

2) The reflecting density function \( g(z) \). It corresponds to the density of the mirror paths, the symmetric process (by the logarithmic distance as explained in the appendix) adjusted by the standard Girsanov accommodation of the drift. In other words it corresponds to the density of the Brownian paths that do touch the barrier.

* See appendix for risk neutrality. Given that the goal is not the pricing of a security the expected return of the security does not need to be risk-neutral.
where $H$ is the barrier level, $S_0$ the initial security price, thus $\log[H/S_0]$ is the threshold negative return accepted by the operator.

We can use the standard proof for barrier options. There exists a probability measure $Q$ equivalent to $P$, under which the random variable $z$ has no drift. We know that the distribution function $G(x)$ can be expressed as the expectation under $P$ of the product of the two indicator functions, one that the market ends up through the barrier and the other that the stopping time $\tau$, defined as the first exit through the barrier, is shorter than $t$. Thus: $E^P[1_{x<z} \cdot 1_{\tau<t}]$, by Girsanov’s theorem, is equivalent to:

\[
(2-4) \quad E^Q[1_{x<z} \cdot 1_{\tau<t} \cdot \exp(-\mu x - \frac{1}{2} \mu^2 t)]
\]

Using the reflection principle for the mirror paths, $(2-4)$ becomes:

\[
(2-5) \quad E^Q[1_{x<2b-z} \cdot \exp\{\mu(2b-x) - \frac{1}{2} \mu^2 t\}]
\]

which, by standard methods (see Taleb, 1997 for a review), yield the above result. QED.

3) The Dirac delta mass of absorption concentrated at the stop loss point $\ln[H/S_0]$, $\delta(.)$, which will correspond to $1$ minus the probability of not hitting the barrier between $t_0$ and $t$.

The final density:

\[
p(z) = f(z) - g(z) \quad \text{for } z > \log(H/S_0)
\]
\[ p(z) = \delta (z - \log(H/S_0)) (1 + G(\log(H/S_0)) - (\log(H/S_0))) \]

for \( z = \log(H/S_0) \)

\[ p(z) = 0 \quad \text{for} \quad z \leq \log(H/S_0) \]

Where \( G(z) \) and \( F(z) \) are the distribution functions of \( g(z) \) and \( f(z) \) respectively defined for \( z \in \) \( \log(H/S_0), \infty \).

**Plot of \( p(z) \):**

*Figure 1: Distribution with different levels of a stop barrier \( A < B < C \).*

**Moment generating function** to the right of the barrier

(2-6) \[ \Phi_r(s) \equiv \int_{\infty}^{\ln H} e^{sx} p(z) dz \]

which obtains

(2-7) \[ \Phi_r(s) = \frac{1}{2} e^{\frac{1}{2}(2s + \sigma^2)} \left( N\left(\beta_1\right) + N\left(\beta_2\left(\frac{H}{S_0}\right)^{2(1+\frac{\mu}{\sigma^2})}\right) - 1 \right) \]

where
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\[ \beta_1 = \frac{1}{\sigma \sqrt{t}} \ln \left( \frac{H}{S_0} \right) - (\mu + s\sigma^2)t \]
\[ \beta_2 = \frac{1}{\sigma \sqrt{t}} \ln \left( \frac{H}{S_0} \right) + (\mu + s\sigma^2)t \]

**Moment generating function** to the left of the barrier

\[ \Phi_l(s) = \int_{-\infty}^{\ln \left( \frac{H}{S_0} \right)} e^{sz} p(z)dz \]

which is equivalent to: \( e^{s \log(H/S_0)}(1-\Phi_l(0)) \), which obtains:

\[ \Phi_l(s) = e^{s \ln \left( \frac{H}{S_0} \right)} \left( 1 + \frac{1}{2} N(\lambda_2) \left( \frac{H}{S_0} \right)^{\frac{2\mu}{\sigma^2}} + N(\lambda_1) - 1 \right) \]

where

\[ \lambda_1 = \frac{1}{\sigma \sqrt{t}} \ln \left( \frac{H}{S_0} \right) + \mu t \]
\[ \lambda_2 = \frac{1}{\sigma \sqrt{t}} \ln \left( \frac{H}{S_0} \right) - \mu t \]

The total moment generating function \( \Phi(s) \) will be the sum of \( \Phi_l(s) \) and \( \Phi_r(s) \). We will next derive the following moments as well as the expected stopping time and compare them to the static unconditional distribution, with means \( \mu \), variance \( \sigma^2 \), skew 0 and kurtosis close to 3 and stopping time equal to nominal time for the strategy.

\[ E(z) = \Phi'(s)|_{s=0} \]

therefore

\[ E(z) = \left( \ln \left( \frac{H}{S_0} \right) \left( N(\lambda_2) - 1 + \left( N(\lambda_1) - 2 \right) \left( \frac{H}{S_0} \right)^{\frac{2\mu}{\sigma^2}} \right) + \mu \left( 2 - N(\lambda_2) + (N(\lambda_1) - 2) \left( \frac{H}{S_0} \right)^{\frac{2\mu}{\sigma^2}} \right) \right) \]

**Variance:** The variance of the portfolio subjected to the stop rules will drop (the second derivative of \( \Phi''(s)|_{s=0} \) yields a clumsy closed form solution).
We will see that it is not quite revealing of the total risks. The standard deviation is shown in Figure 2.

Figure 2 Standard Deviation as a function of the stop-loss level, using $\mu=0$, $S_0=100$, $\sigma=.157$ (annualized, hence .00825 daily), $t=1$ day ($1/365$).

Stopping Time analysis: $E(\tau)$ is the expected stopping time $\tau$, when the returns $z$ reach the liquidation level $z^*$. Using conventional methods, the density of stopping time can be computed as follows. Take $\tau$ the stopping time

$$\tau = \inf \left\{ \tau; S_\tau > \ln \left( \frac{H}{S_0} \right) \right\}$$

By standard methods we know the results of the density function with drift:

taking

$$l = \frac{-\mu}{\sigma} \frac{\sigma}{2}$$

and

$$h = \frac{1}{\sigma} \ln \left( \frac{H}{S_0} \right)$$

(2.12) $$\phi(\tau) = \frac{h}{\sqrt{2\pi\tau^3}} e^{-\frac{h^2}{2\tau}}$$
We remark that the frequency of the losses for the investor will increase as a function of the proximity of the stop loss. This illustrates the fallacy of frequency-based track records. In other words, the analysis by frequency can lead to distortions (as with every asymmetric random variable).

Over period $t$, the probability of incurring a loss $P[z<0]$ will be $\int_{\ln\left(\frac{H}{S_0}\right)}^{0} p(x)dx$.

The probability of deriving a profit declines as the stop loss point becomes closer to the present asset price—without impacting the Sharpe ratio (hence the limitations to the two-moment dimension of the Sharpe ratio).

Define the probability of hitting the barrier $\pi$. It is the $P[\tau<t]$, which can also be calculated as $\pi = \int_{\ln\left(\frac{H}{S_0}\right)}^{0} p(x)dx$, hence

\begin{equation}
\pi = 1 - G\left(\ln\left(\frac{H}{S_0}\right)\right) - F\left(\ln\left(\frac{H}{S_0}\right)\right)
\end{equation}

The intuition of the skewness can be shown through analysis of the asymmetry of the probability of profit (conditional on a constant Sharpe ratio).
Figure 4 Probability of Hitting the barrier assuming \( S_0 = 100, t = 1/365, \sigma = .157 \). The vertical axis displays the probability of hitting the stop loss while the horizontal one shows the price at which the stop loss is placed.

Figure 5 Probability of deriving a profit over \( t \), assuming \( S_0 = 100, t = 1/365, \sigma = .157 \). The vertical axis displays the probability while the horizontal one shows the price at which the barrier is placed.

As to transaction costs, they correspond to the expected probability of hitting the barrier times the costs incurred. A stop order is defined as counter to the limit order from a microstructure standpoint, where I argued in Taleb (1997) that the market maker incurs no transaction costs. Here the stop loss belongs to the category of stop orders, which makes transaction costs apply in full. Using the following proportional transaction costs framework:

\[
S_t \begin{cases} 
S_t \left(1 + \frac{k}{2}\right) & \text{when buying} \\
S_t \left(1 - \frac{k}{2}\right) & \text{when selling}
\end{cases}
\]
The operator will incur an additional cost of $- k S_t/2$ at the lower barrier, here $k H/2$, with probability $\pi$. Therefore the expectation of wealth will simply be shifted down by

\[
(2-14) \quad - \pi \frac{H k}{2}
\]

which corresponds to the expectation weighted by stopping time plus the expectation of the charge at the barrier.

The trade-off between the expectation of the costs and the location of the stop-loss can be illustrated as follows, in Figure 6.

*Figure 6* Expected transaction costs at the stop level assuming proportional transaction costs represent .2% of asset value at the stop.

### 2.1.2 Behavior Under Temporal Aggregation

$\Phi(s)$ aggregates into $\Phi(s)^n$ as we add the trading days, with the results that we converge at the end to a simple symmetric Gaussian with lower variance than otherwise—but possibly a lower mean in the presence of transaction costs. To prove this is suffices to see the behavior of the density function under Laplace inversion or the behavior of the third moment under temporal aggregation. Figure 7 plots a numerical example.

*Figure 7* Skew under aggregation. The vertical axis displays the third (centered) moment while the horizontal one shows the number of successive aggregations.