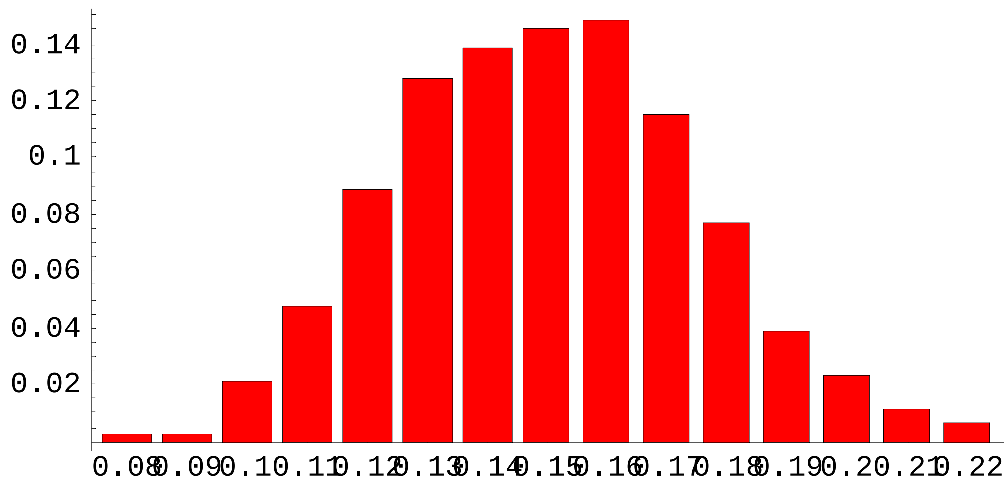
VOLATILITY IS NOT AS STOCHASTIC AS YOU THINK!

It came to my notice that participants overreact to changes in historical volatility. They read too much in its variations. This note discusses some counterintuitive properties of stochastic volatility. I distinguish between the *native stochastic volatility* that corresponds to a normal variation that results from sampling, and the true stochastic volatility that results from the changes in the characteristics of the distribution or its parameters.

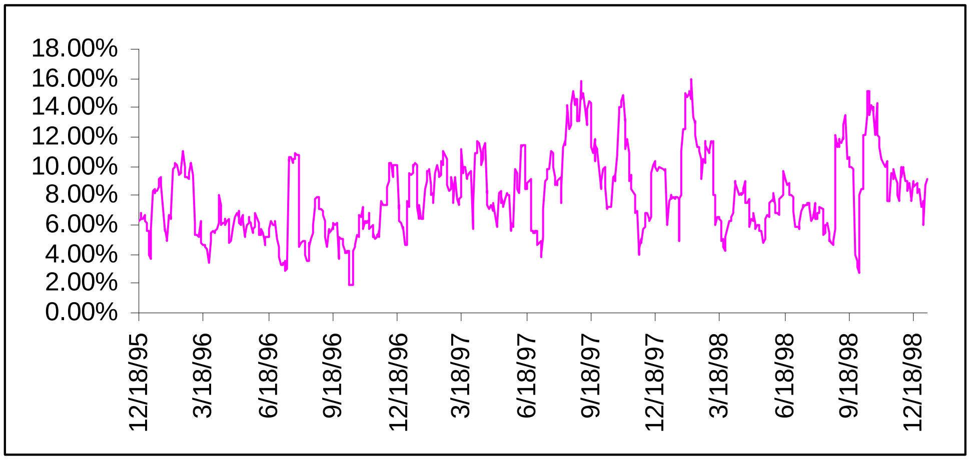
Assume that markets are normally distributed. The volatility over a n- period (say one month) will not be delivered constant. It will change according to the sample. If we try a Monte Carlo experiment and generate a Brownian motion with the volatility of the logarithms of the changes σ=15.7%, the following dispersion in the observed standard deviation occurs:

Figure 1 Distribution of Observed Volatility when Volatility is Constant



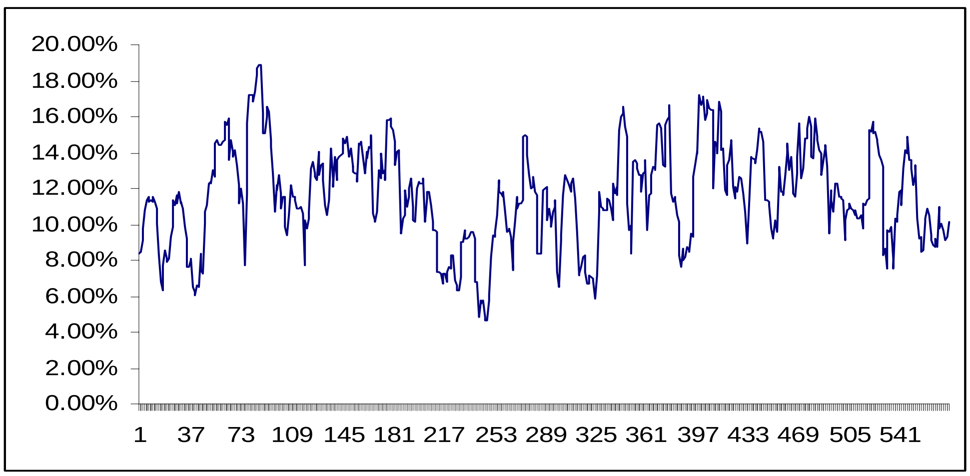
The next graph shows the times series of the rolling historical USD-DEM 10 business day volatility.

Figure 2 Rolling 10-day USD-DEM Volatility



The next one is the rolling 10-day volatility of a Monte Carlo sampling from a pure constant volatility process.

Figure 3 Rolling 10-day Volatility For a Constant Volatility Asset



The next step will allow us to formalize this intuition and derive a quantitative method to exploit the issue.

TECHNICAL DERIVATIONS

Distribution of the Variance

We cannot immediately derive the distribution of the standard deviation; we will initially work with that of the variance. Take the return $x \sim N(\mu,\sigma)$; assume $\mu=0$ without any loss of generality. We try first to generate the distribution of x^2. If x has density f then $y = f(x)$ has density

(1.) $$\frac{f(g(y))}{f'(g(y))}$$

where *g(.)* is the inverse function, *g(y)=x* .
Here $y=x^2$ but we have the complication here of having

$x = g(y) = \sqrt{y}$ if $x>0$ and
$x = g(y) = -\sqrt{y}$ if $x \le 0$

Hence the probability density function is the sum of both, $\frac{N(\sqrt{y}) + N(-\sqrt{y})}{2\sqrt{y}}$

Hence

(2.) $$g(y) = \frac{1}{2\sigma\sqrt{2\pi\, y}} e^{\frac{-y}{2\sigma^2}}$$

Which is the special case of the general gamma distribution, as $G(x|\alpha,p) = \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}$ ($\alpha>0$, $p>0$) (here $\alpha=(2\sigma^2)^{-1}$, $p= ½$).

The characteristic function is $\Phi(t) = \frac{1}{\sqrt{1-2it\sigma^2}}$, hence the expectation can be derived as σ^2 and the variance $2\sigma^4$.

Summing x^2 we have (by convolution) the distribution of $y' = \sum_{i=1}^{n} y_i$ a gamma distribution,

(3.) $$G(y' |(2\sigma^2)^{-1}, n/2) = \frac{2^{-n/2} e^{-\frac{y'}{2n\sigma^2}} y'^{\frac{n}{2}-1} \sigma^{-n}}{\Gamma(n/2)}$$

(for which a $\sigma=1$ corresponds to a Chi-square distribution for y/n with n degrees of freedom). We get the distribution of the average using (1) again, by deriving the distribution of $z \equiv y'/n$
hence

(4.) $$G(z) = \frac{2^{-n/2} e^{-\frac{nz}{2\sigma^2}} (nz)^{\frac{n}{2}} \sigma^{-n}}{z\Gamma(n/2)}$$

with characteristic function

$$\Phi(t) = n^{n/2}\left(-2\,i\,t + \frac{n}{\sigma^2}\right)^{-n/2} \sigma^{-n}$$

The expectation and variance of the *average* (i.e. the observed variance) become σ^2 and $2\sigma^4/n$.

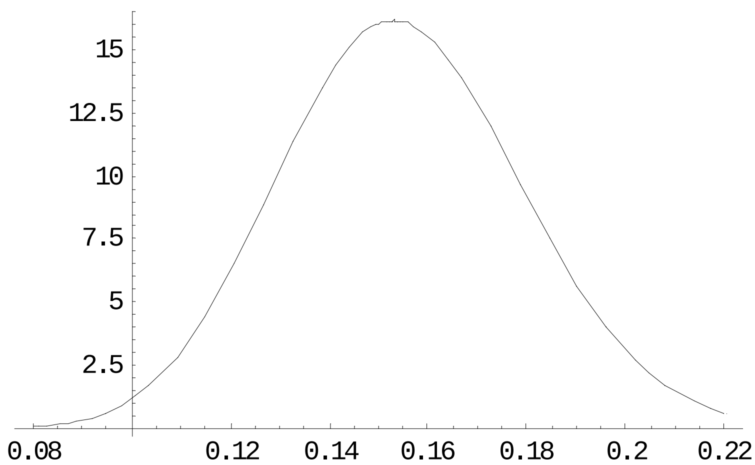
Distribution of the Standard Deviation.

Moving to the standard deviation, that is x ≡√z, we repeat the previous attempt, but in reverse to recover the density function of concern. As we have the new density defined in an area that is strictly positive, we get the new one p(.) as 2 g(x^2)x, hence

$$P(x)=\frac{2^{1-\frac{n}{2}}\, e^{\frac{-nx^2}{2\sigma^2}}\, n^{n/2}\; x^{n-1}\sigma^{-n}}{\Gamma[n/2]}$$

Where z is the Square root of the sums of the variation of normal variate, and n the number of observations. We next recover the Figure 1 by plotting p[x]. .

Figure 2 Density of the expected volatility, with n=20 days, σ=15.7%



The characteristic function becomes

$$\frac{1}{2\,\Gamma\left(\frac{n}{2}+1\right)}\left(n^{n/2}\,(\sigma^2)^{-\frac{n}{2}}\left(\frac{\sigma^2}{n}\right)^{\frac{n-1}{2}}\left(\sqrt{2}\,i\,t\,\Gamma\left(\frac{n+1}{2}\right)\,{}_1F_1\left(\frac{n+1}{2};\,\frac{3}{2};\,-\frac{t^2\sigma^2}{2n}\right)\sigma^2+ 2\sqrt{\frac{\sigma^2}{n}}\;\Gamma\left(\frac{n}{2}+1\right)\,{}_1F_1\left(\frac{n}{2};\,\frac{1}{2};\,-\frac{t^2\sigma^2}{2n}\right)\right)\right)$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function.
Deriving the characteristic function, we thus recover the expected mean, here

$m=\dfrac{\sqrt{2}\,\sigma\,\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n}\quad\Gamma\left(\frac{n}{2}\right)}$, which , for n large, can be approximated by σ (to no greatsurprise),

since

$$\lim_{n->\infty} \frac{\sqrt{2}\ \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n}\quad \Gamma\left(\frac{n}{2}\right)} = 1$$

The variance becomes $\sigma^2\left(1 - \frac{n\ \Gamma\left(\frac{n+1}{2}\right)^2}{2\Gamma\left(1+\frac{n}{2}\right)^2}\right)$.

The following is worth noting: As the observation period lengthens, $\frac{n\ \Gamma\left(\frac{n+1}{2}\right)^2}{2\,\Gamma\left(\frac{n}{2}\right)^2}$ goes to 1 and the spurious "volga" (i.e. the volatility of volatility) asymptotically decreases to 0.

WHAT IS NORMAL VOLATILITY OF HISTORICAL VOLATILITY?

Assuming σ=10%, with daily observation period, the following standard deviations of the distribution of volatility can be expected.

Duration	**Vvol**
2 weeks	2.2%
1 month	1.5%
2 months	1.1%
3 months	.9%
6 months	.57%
1 year	.4%
2 years	.23