Poisson Buster & Evidence of Scalability

Or Why "Fat Tails" are not Poisson

Nassim Nicholas Taleb

This brief test using 12 million pieces of exhaustive returns shows how equity prices (as well as short term interest rates) do not have a characteristic scale. No other possible method than a Paretan tail, albeit of unprecise calibration, can charaterize them.

Introduction: This is a quick technical note to *The Black Swan* and my article in *The American Statistician*. It aims to supply convincing evidence of scalability and NonPoisson-ness of the data --assuming a standard Poisson. Thanks to the need for the probabililities add up to 1 (something even economists seem to agree with), scalability in the tails is the sole possible model for such data. We may not be able to write the model for the full distribution --but we know how it looks like in the tails, where it matters.

The Behavior of Conditional Averages: With a scalable (or "scale-free") distribution, when K is "in the tails" (say you reach the point when $f[x, \alpha] = C X^{-\alpha}$, where C is a constant and α the power law exponent), the relative conditional expectation of X (knowing that $x>K$) divided by K, that is, $\frac{E[X|X>K]}{K}$ is a constant, and does not depend on K. More precisely, it is $\frac{\alpha}{(\alpha-1)}$.

$$\frac{\int_K^\infty x\, f(x, \alpha)\, dx}{\int_K^\infty f(x, \alpha)\, dx} = \frac{K\, \alpha}{\alpha - 1}$$

This provides for a handy way to ascertain scalability by raising K and looking at the averages in the data.
Note further that, for a standard Poisson, (too obvious for a Gaussian): not only the conditional expectation depends on K, but it "wanes", i.e.

$$\underset{K\to\infty}{\text{Limit}} \left(\frac{\int_K^\infty \frac{m^x}{\Gamma(x)}\, dx}{\int_K^\infty \frac{m^x}{x!}\, dx} \right) \Big/ K = 1$$

Other Decompositions: The result of course would cancel the kind of representations such as the model called Duffie-Pan-Singleton, which decomposes generating processes into a sum of jumps and some diffusion. Unless they have an infinity of power-law jumps, the conditional average would lose its scalability *beyond* the worst jump.

Calibrating Tail Exponents. In addition, we can calibrate power laws. Using K as the cross-over point, we get the α exponent above it --the same as if we used the Hill estimator or ran a regression above some point.

Leave it to the Data

Data: Pallop Angsupun from **Universa** ran the following test: We collected the most recent 10 years of daily prices for stocks (no survivorship bias effect as we included companies that have been delisted up to the last trading day), n= 11,674,825 , deviations expressed in logarithmic returns.
We scaled the data using various methods.

1- In the "sigma space" test we used a rolling 22 day window scaled by the noncentral standard deviations. We did not add a mean for reasons explained elsewhere (Taleb, revised *Dynamic Hedging* forthcoming 2008). So for every asset *i,*

$$\frac{\text{Log}\,\frac{S_t{}^i}{S_{t-1}{}^i}}{S_{t-1}\sqrt{\sum_{j=0}^{22}\text{Log}\left(\frac{S_{t-j}{}^i}{S_{t-j-1}{}^i}\right)^2}}$$

The expression in "numbers of sigma" or standard deviations is there to conform to industry language (it does depend somewhat on the stability of sigma).

2- In the "MAD" space test we used the mean deviation.

$$\frac{\text{Log}\,\frac{S_t{}^i}{S_{t-1}{}^i}}{\left|\sum_{j=0}^{22}\text{Log}\left(\frac{S_{t-j}{}^i}{S_{t-j-1}{}^i}\right)^2\right|}$$

We focused on negative deviations. For instance,in the Table x below,the average move below "7 standard deviations",-7,is 11.20 standard deviations,that is a multiple of 1.60.We kept moving K up until to 100 sigmas (indeed) --and we still had observations.
Note the tail estimator

$$\text{Implied}\ \alpha\,|_K = \frac{E[X\ |_{X>K}]}{E[X\ |_{X>K}] - K}$$

-

- **Sigma-Space Table**

```
TableForm[mvsig2, TableHeadings → {None,
   {"Sigma", "E[X|X>K]", "n", "ratio E[X|X>K]/K", "Implied α"}}]
```

Sigma	$E[X\|_{X>K}]$	n	ratio $\frac{E[X\|_{X>K}]}{K}$	Implied α
-1.	-1.74525	1.5242×10^{6}	1.74525	2.34183
-2.	-3.01389	343952.	1.50695	2.9726
-3.	-4.58148	99404.	1.52716	2.89696

-4.	-6.30102	40525.	1.57525	2.73836
-5.	-8.02031	21156.	1.60406	2.65546
-6.	-9.61995	13058.	1.60333	2.65748
-7.	-11.1952	8720.	1.59932	2.66857
-8.	-12.6746	6245.	1.58432	2.71139
-9.	-14.1636	4613.	1.57373	2.74297
-10.	-15.6078	3528.	1.56078	2.78324
-11.	-17.0819	2742.	1.5529	2.80865
-12.	-18.4656	2199.	1.5388	2.85597
-13.	-19.9193	1770.	1.53226	2.87879
-14.	-21.2999	1458.	1.52142	2.91784
-15.	-22.6279	1219.	1.50853	2.96647
-16.	-23.9772	1024.	1.49858	3.00571
-17.	-25.4426	856.	1.49662	3.01359
-18.	-26.8507	728.	1.4917	3.03374
-19.	-28.7366	594.	1.51245	2.9514
-20.	-30.4193	503.	1.52096	2.91952
-25.	-39.7594	195.	1.59038	2.69383
-30.	-50.6481	105.	1.68827	2.45292
-35.	-64.9988	59.	1.85711	2.16671
-40.	-82.8163	36.	2.07041	1.93422
-45.	-98.1439	26.	2.18098	1.84676
-50.	-113.324	20.	2.26649	1.78958
-55.	-128.696	16.	2.33993	1.74631
-60.	-152.754	12.	2.54591	1.64687
-65.	-160.867	11.	2.47487	1.67802
-70.	-170.105	10.	2.43007	1.69927
-75.	-180.84	9.	2.4112	1.70861
-80.	-180.84	9.	2.2605	1.79333
-85.	-180.84	9.	2.12753	1.88689
-90.	-192.543	8.	2.13937	1.87768
-95.	-226.232	6.	2.38139	1.72391
-100.	-251.691	5.	2.51691	1.65923

-

- **MAD-Space Table**

```
table2 = TableForm[mvsig2, TableHeadings → {None,
   {"Sigma", "E[X|X>K]", "n", "ratio E[X |X>K]/K", "Implied α"}}]
```

Sigma	$E[X\|_{X>K}]$	n	ratio $\frac{E[X\|_{X>K}]}{K}$	Implied α
-1.	-1.75202	1.32517×10^{6}	1.75202	2.32974
-2.	-3.02395	300806.	1.51197	2.95322
-3.	-4.59956	87788.	1.53319	2.87552
-4.	-6.30478	36290.	1.57619	2.73552
-5.	-7.96354	19285.	1.59271	2.68717
-6.	-9.55642	11818.	1.59274	2.68709

-7.	-11.0487	7985.	1.57839	2.72893
-8.	-12.5138	5668.	1.56423	2.77233
-9.	-13.9613	4174.	1.55126	2.81403
-10.	-15.3283	3198.	1.53283	2.87678
-11.	-16.6493	2515.	1.51357	2.94715
-12.	-18.0656	1976.	1.50547	2.97837
-13.	-19.4719	1578.	1.49784	3.00869
-14.	-20.8333	1286.	1.48809	3.04879
-15.	-22.3211	1042.	1.48807	3.04888
-16.	-23.8607	851.	1.49129	3.03545
-17.	-25.3371	709.	1.49041	3.03909
-18.	-27.088	580.	1.50489	2.98064
-19.	-28.6489	491.	1.50783	2.96915
-20.	-30.2472	418.	1.51236	2.95176
-25.	-40.8788	181.	1.63515	2.57443
-30.	-51.0157	103.	1.70052	2.42751
-35.	-66.7369	56.	1.90677	2.10282
-40.	-79.4406	39.	1.98602	2.01418
-45.	-92.205	29.	2.049	1.95329
-50.	-101.755	24.	2.0351	1.96609
-55.	-121.808	17.	2.2147	1.82325
-60.	-148.755	12.	2.47925	1.67602
-65.	-156.709	11.	2.41091	1.70876
-70.	-156.709	11.	2.23871	1.80729
-75.	-175.422	9.	2.33896	1.74685
-80.	-203.991	7.	2.54988	1.64521
-85.	-203.991	7.	2.39989	1.71434
-90.	-203.991	7.	2.26656	1.78954
-95.	-203.991	7.	2.14727	1.87164
-100.	-203.991	7.	2.03991	1.96163

Short term Interest Rates

■ EuroDollars Front Month 1986-2006

n=4947

MAD	$E[X\|_{X>K}]$	n	ratio $\frac{E[X\|_{X>K}]}{K}$	Implied α
-0.5	-1.8034	1520	3.6068	1.38361
-1.	-2.41323	969	2.41323	1.7076
-1.5	-3.09923	613	2.06616	1.93795
-2.	-3.81882	401	1.90941	2.09961
-2.5	-4.44704	287	1.77881	2.284
-3.	-5.16202	203	1.72067	2.38759
-3.5	-5.84771	150	1.67077	2.49081
-4.	-6.81986	103	1.70497	2.41851
-4.5	-7.36791	85	1.63731	2.56909

-5.	-7.96752	69	1.5935	2.68491
-5.5	-8.923	51	1.62236	2.60678
-6.	-9.2521	46	1.54202	2.84496
-6.5	-9.86946	38	1.51838	2.92909
-7.	-10.2338	34	1.46197	3.16464
-7.5	-11.0244	27	1.46992	3.128
-8.	-11.4367	24	1.42959	3.32782

■ UK Rates 1990-2007

n=4143

MAD	$E[X\|_{X>K}]$	n	ratio $\frac{E[X\|_{X>K}]}{K}$	Implied α
0.5	1.68802	1270	3.37605	1.42087
1.	2.23822	806	2.23822	1.80761
1.5	2.94721	474	1.96481	2.03648
2.	3.56121	315	1.7806	2.28106
2.5	4.22146	211	1.68858	2.45225
3.	4.97319	140	1.65773	2.52038
3.5	5.64241	101	1.61212	2.63368
4.	6.51723	69	1.62931	2.58905
4.5	7.40975	50	1.64661	2.54653
5.	8.43269	36	1.68654	2.45658
5.5	8.95114	31	1.62748	2.59368
6.	9.56132	26	1.59355	2.68477
6.5	10.1677	22	1.56426	2.77224
7.	11.4763	16	1.63947	2.56381

Literally, you do not even have a large number K for which scalability drops from a small sample effect.

■ USD-JPY (1971-2007) (Negative Domain)

MAD	$E[X\|_{X>K}]$	n	ratio $\frac{E[X\|_{X>K}]}{K}$	Implied α
-1	-2.14951	1674	2.14951	1.86993
-1.25	-2.39811	1349	1.91849	2.08874
-1.5	-2.68188	1058	1.78792	2.26916
-1.75	-2.98844	822	1.70768	2.41306
-2.	-3.29019	648	1.64509	2.55016
-2.25	-3.56329	526	1.58368	2.71326
-2.5	-3.85498	423	1.54199	2.84505
-2.75	-4.10008	353	1.49094	3.03691
-3.	-4.38008	288	1.46003	3.17378
-3.25	-4.60737	244	1.41765	3.39433
-3.5	-5.0187	183	1.43392	3.3046
-3.75	-5.30525	152	1.41473	3.41119
-4.	-5.69791	119	1.42448	3.35584
-4.25	-6.04265	98	1.4218	3.3708
-4.5	-6.25078	87	1.38906	3.57029
-4.75	-6.59176	72	1.38774	3.57906
-5.	-6.74883	66	1.34977	3.85906
-5.25	-7.01339	57	1.33588	3.97723
-5.5	-7.20808	51	1.31056	4.21999
-5.75	-7.50068	43	1.30447	4.28444
-6.	-7.92747	34	1.32125	4.11288
-6.25	-8.47831	26	1.35653	3.80482
-6.5	-8.97441	21	1.38068	3.62689
-6.75	-9.2219	19	1.36621	3.7307
-7.	-9.66084	16	1.38012	3.63075
-7.25	-10.8153	11	1.49177	3.03347
-7.5	-11.15	10	1.48667	3.05479
-7.75	-11.15	10	1.43871	3.2794
-8.	-12.5366	7	1.56708	2.76342
-8.25	-13.2717	6	1.60869	2.64286
-8.5	-13.2717	6	1.56138	2.78133
-8.75	-13.2717	6	1.51677	2.9351

We get scalability as far as meets the eye. Usually small sample effects cause us to not observe much of the tails, with the consequence of "thinning" the upper bound. We do not even witness such effect.

Longer Windows

A longer window, by taking time-aggregates, such as weeks, and months, do not show any different result --which is an additional evidence of the failure of Poisson. For instance weekly tails exhibit *thickening* instead of flattening: the implied α drops!
Clearly this can come from the lack of independence. But it can also result from the slowness of the convergence to a Gaussian.

```
table3 = TableForm[mvsig2, TableHeadings → {None,
    {"Sigma", "E[X|X>K]", "n", "ratio E[X|X>K]/K", "Implied α"}}]
```

Sigma	$E[X\|_{X>K}]$	n	ratio $\frac{E[X\|_{X>K}]}{K}$	Implied α
-1.	-1.71473	270506.	1.71473	2.39914
-2.	-2.85916	61149.	1.42958	3.32785
-3.	-4.16075	16568.	1.38692	3.58453
-4.	-5.49431	6030.	1.37358	3.67682
-5.	-6.88222	2638.	1.37644	3.65644
-6.	-8.37999	1296.	1.39667	3.52102
-7.	-10.0134	700.	1.43048	3.32299
-8.	-11.4757	446.	1.43446	3.30172
-9.	-13.4347	272.	1.49274	3.02947
-10.	-15.1321	190.	1.51321	2.9485
-11.	-17.8609	120.	1.62372	2.60329
-12.	-20.6359	84.	1.71966	2.38955
-13.	-25.18	54.	1.93692	2.06733
-14.	-27.5105	45.	1.96504	2.03623
-15.	-31.7716	34.	2.11811	1.89437
-16.	-33.9109	30.	2.11943	1.89331
-17.	-39.2356	23.	2.30798	1.76454
-18.	-42.4653	20.	2.35918	1.73574
-19.	-46.7196	17.	2.45893	1.68544
-20.	-52.5833	14.	2.62916	1.61381