Derivatives, Prediction and *True* Fat Tails (i.e. Fractal), Part 1: The Fragility of Option Pricing

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This is a working notebook --it cannot be quoted in this present version.

Derivatives that depends on the high consequential large deviation are marred with huge sampling error. I examine the sensitivity of the derivatives to the parameters, the sampling error of the estimations or "predictions", then look at the empirical stability of these parameters.
Organization: First 1) I do the math of distribution & derivatives, as there is no intelligent literature on the subject outside of the inapplicable Levy-Stable, 2) I show the magnitudes errors w.r.t. to some parameters (mainly the tail exponent α) 3) I discuss the error in the estimation of these parameters.
Main point: For options on remote events, a small change in the tail exponent say α betwen 1.5 and 2, well within the estimation errors, make the option change in value: a .5 change in exponent makes the error on the event vary by a factor >10, often >100. Moral: don't play with tail estimations, and don't believe that options can estimate anything.

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1- True Fat Tails and Derivatives Pricing

Definition: true fat tails (see lecture x) are as follows $P_{>nX}/P_{>X}$ depends on n, not X for X large enough.

First, we select a distribution without a tail-characteristic scale for x on the real line -∞ and ∞, which consists in a fractal tails with exponent α and a multiplying scale. Typical Student T

$$\phi\,(u) = \frac{1}{\sqrt{\alpha}\;\beta\left(\frac{\alpha}{2},\frac{1}{2}\right)}\left(\frac{\alpha}{\alpha+u^2}\right)^{\frac{1+\alpha}{2}},\quad u\in[-\infty,\infty],\ \alpha\geq 1$$

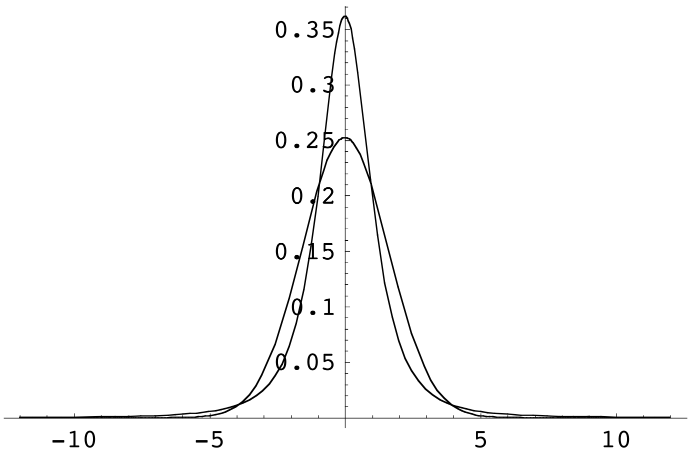
So for large u "in the tails", we can see that it behaves K $u^{-\alpha-1}$, where K is a constant.

Where β (.) is **Euler beta function** $\beta\,(a,\ b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}\,(1-t)^{b-1}\,dt.$

ϕ(.) has fractal tails with exponent α on both sides.
Note: I ignore the designation "Levy-stable"

α= 5/2; Comparison with a Gaussian N(0,$\sqrt{5/2}$)

Comparison with an equivalent Gaussian



Now consider the multiplicative (monoperiodic) process X= X_0 (1+s u -c), where u is ϕ distributed. $\frac{X-X0-cX0}{X0}\,\frac{1}{s}$ is a straight relative price change with a "drift" term c and a "dispersion" constant s to scale by the "volatility", simplified as a multiple of mean deviation (for a given period between an initial 0 and T.) The problem is that we cannot take a fractal tailed distribution for Log[$\frac{X}{X_0}$] for obvious reasons (too unwieldy; I tried), so we have to be content with relative price changes.

By change of stochastic variables, I am able to get the distribution of X, conditional on X_0.

(If x has distribution f then y=z(x) has density $\frac{f(g(x))}{f'(g(x))}$ where g is the inverse function of z).

$$f(X) = \frac{1}{\sqrt{\alpha}\; s\, X_0\;\; \beta(\frac{\alpha}{2},\, \frac{1}{2})} \left(\frac{\alpha}{\frac{(X-C\,X_0-X_0)^2}{s^2\; v^2} + \alpha} \right)^{\frac{\alpha+1}{2}},\;\; X > 0$$

Caveat 1 and Renormalization: The distribution *f(X)* may have minutely small mass for *X*<0, when *(1+s u)* turns negative, *s u*< *1*. This requires an atrocially huge volatility and can be compensated by a truncating effect and renormalization of the mass with f'[x]= f(x) $\frac{1}{1-\int_{-\infty}^{0} f(X)\, dX}$. I left it out as it does not affect the exercise.

Indeed

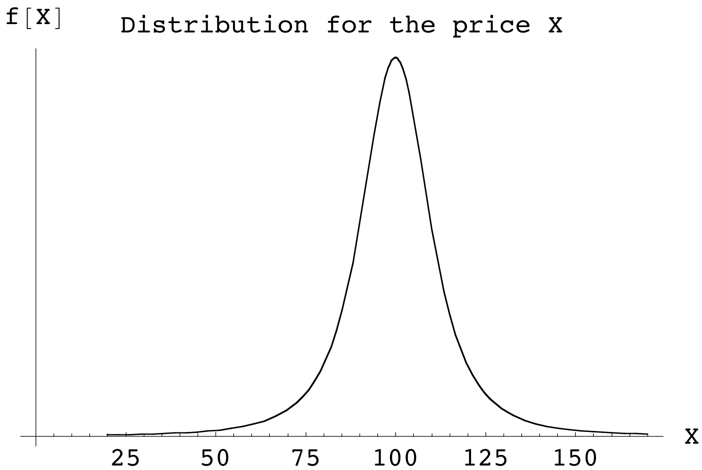
$$\int_{-\infty}^{0} f(X) \Big|_{\alpha=2} \, dX = \frac{1}{2} - \frac{1}{2\sqrt{1+2 s^2}}$$

and

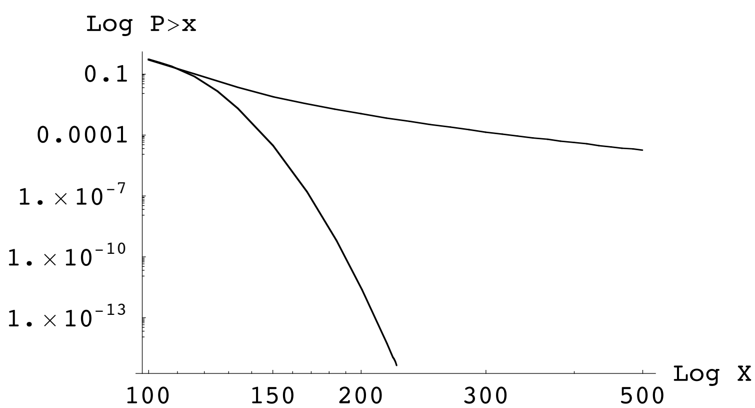
$$\int_{-\infty}^{0} f(X) \mid_{\alpha=3} \, dX \, (6 s^2 + 2) \sin^{-1}\left(\frac{\sqrt{\sqrt{3 s^2+1}\, -1}}{\sqrt{2}\; \sqrt[4]{3 s^2+1}} \right) - \frac{\sqrt{3}\; s}{3 \pi s^2 + \pi}$$

Which is very small: of the order of < .01% for high volatility environments when α=3, and .05% when α=2 --thus justifying ignoring the renormalization.

Caveat 2 and Explosive Mean: Likewise the mean may become explosive upwards, in which case the compensation can be part of the drift c just like the lognormal is compensated by a negative $\frac{-1}{2}\sigma^2$ (where σ is the Gaussian standard deviation. But,for the purposes of the exercise f(.) works well in addressing option errors.



Scalable as far as meets the eye (unlike the Lognormal)



I was not able to find a close solution except for α=2,3. At α=∞ we get the standard Bachelier-Thorp (a.k.a. Black-Scholes) equation.

Note on moments: With no drift *c*,

Whith finite variance α=3

$\alpha = 3,$

$$E[X] = X_0 \left(\frac{2\sqrt{3}\, s + 2\cot^{-1}\left(\sqrt{3}\, s\right) + \pi}{2\pi} \right) \simeq X_0$$

where $\cot^{-1}(z)$ is the Arc Cotangent of z

$$E\left[\left(\frac{X-X0}{X0}\right)^2\right]=\frac{3 s^2}{2}$$

$$E\left(\left|\frac{X-X0}{X0}\right|\right)=\frac{\sqrt{3}\, s\,(2+3 s^2)}{\pi+3\pi s^2}$$

With infinite variance (borderline) α=2

$$\alpha=2,\ E[X]=\frac{X_0\left(s+\sqrt{2 s^4+s^2}\right)}{2 s}\simeq X_0$$

$$E\left(\left|\frac{X-X0}{X0}\right|\right)=s\left(\sqrt{2}-\frac{s}{\sqrt{1+2 s^2}}\right)$$

■ Call Options Under Different Parametrizations

a- Call Option Price C with a Cubic α

$$\text{Call Price } C=\int_K^\infty (X-K)\, f(X)\, dX\,\Big|\,\alpha=3$$

$$C_3=\frac{s\,X0\left(\pi\sqrt{\frac{1}{s^2\,X0^2}}\,(-K+C\,X0+X0)+2\sqrt{3}\right)+2\,(-K+C\,X0+X0)\cot^{-1}\left(\frac{\sqrt{3}\,s\,X0}{-K+C\,X0+X0}\right)}{2\pi}$$

I apologize for the inelegance but I can't do better

b- Call Option Price with α= 5/2

$$C_{5/2}=\frac{1}{6\,5^{3/4}\sqrt{\pi}\,\Gamma(\frac{5}{4})}\left(\left(2\sqrt{2}\left(5\sqrt{s\,X0}\,\sqrt[4]{5 s^2\,X0^2+2\,(-K+C\,X0+X0)^2}+\frac{\sqrt[4]{5}\,s\,X0\,(-K+C\,X0+X0)^2\,(\zeta1+5)\,\zeta2}{5 s^2\,X0^2+2\,(-K+C\,X0+X0)^2}\right)+\frac{5^{3/4}\sqrt{\pi}\,(-K+C\,X0+X0)\,\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)\Gamma\left(\frac{7}{4}\right)\right)$$

$$\zeta1=\frac{2\,(-K+C\,X0+X0)^2}{s^2\,X0^2}$$

$$\zeta2={}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{3}{2};-\frac{2\,(-K+C\,X0+X0)^2}{5 s^2\,X0^2}\right)$$

where ${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}(a)_k\,(b)_k\,/\,(c)_k\,z^k\,/\,k!$.

c- Call Option Price with square α

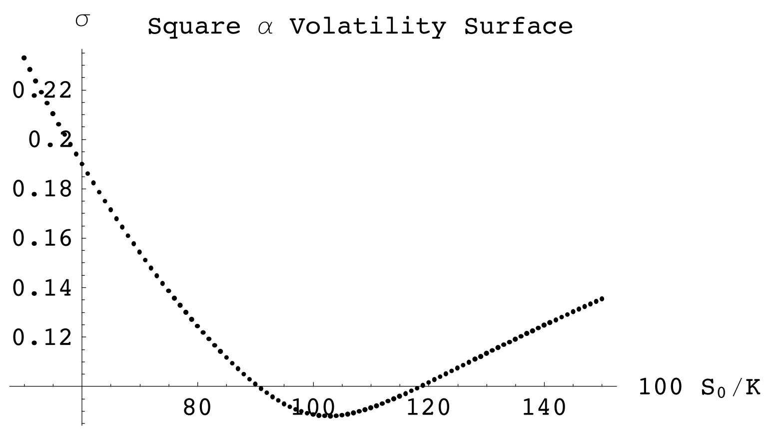
$$C_2 = \frac{1}{2}\left(\zeta 3\, K^2 + (-2\,(C+1)\,\mathrm{X0}\,\zeta 3 - 1)\, K + \mathrm{X0}\,(C + ((C+1)^2 + 2\, s^2)\,\mathrm{X0}\,\zeta 3 + 1)\right)$$

where

$$\zeta 3 = \sqrt{\frac{1}{K^2 - 2\,(1+C)\,K\,\mathrm{X0} + ((1+C)^2 + 2\, s^2)\,\mathrm{X0}^2}}$$

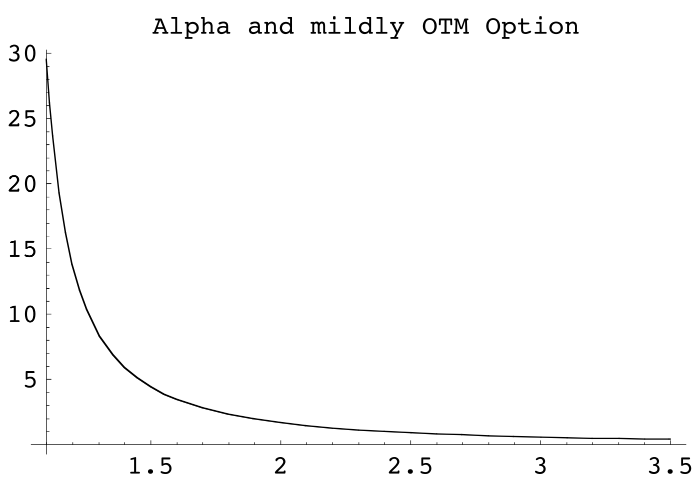
■ Comparison to the Volatility Smile (Bachelier-Thorp, a.k.a. Black Scholes)

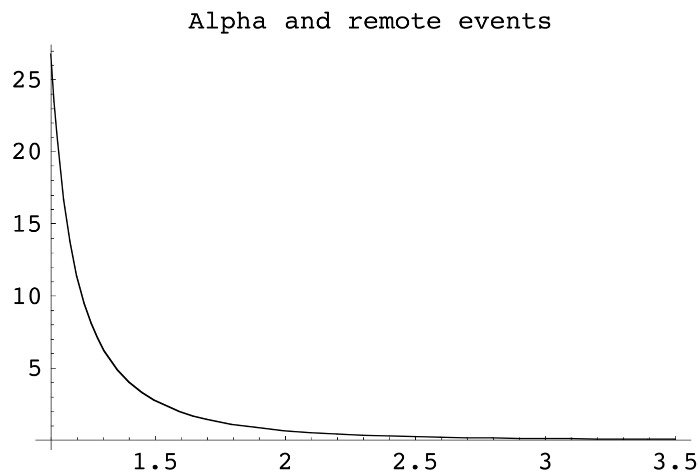
The Infinite Variance Case: $\alpha \leqslant 2$ does not mean anything for option pricing, it generates a volatility surface --so long as the scaling s is calibrated on the absolute first moment.



2- *True* Fat Tails and Derivatives Errors

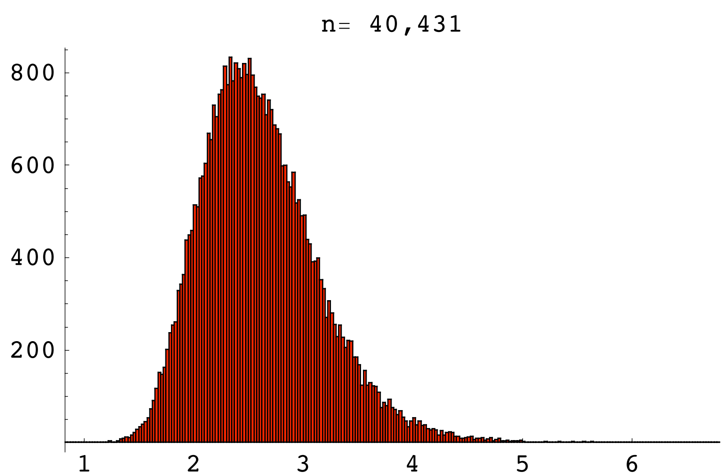
By changing α and maintaining the rest constant, we can do guess the consequence of a small error in the tail exponent on the option value.





Note that almost all options converge to the same price (minus moniness) when alpha drops to close to 1.

Errors in Alpha Estimation



The Mean Deviation= .42 for an estimated alpha of 2.62 (using the Hill Estimator).