Risk Neutral Option Pricing With Neither Dynamic Hedging nor Complete Markets, A Measure-Theoretic Proof

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***Abstract*—Proof that under simple assumptions, such as constraints of Put-Call Parity, the probability measure for the valuation of a European option has the mean of the risk-neutral one, under any general probability distribution, bypassing the Black-Scholes-Merton dynamic hedging argument, and without the requirement of complete markets. We confirm that the heuristics used by traders for centuries are both more robust and more rigorous than held in the economics literature.**

I. Background

Option valuations methodologies have been used by traders for centuries, in an effective way (Haug and Taleb, 2010). In addition, valuations by expectation of terminal payoff center the probability distribution around the "risk-neutral" forward, thanks to Put-Call Parity. The Black Scholes argument (Black and Scholes, 1973, Merton, 1973) is held to allow risk-neutral option pricing thanks to dynamic hedging. This is a puzzle, since: 1) Dynamic Hedging is not operationally feasible in financial markets owing to the dominance of portfolio changes resulting from jumps, 2) The dynamic hedging argument doesn't stand mathematically under fat tails, as it requires a "Black Scholes world" with many impossible assumptions, one of which requires finite quadratic variations, 3) We use the same Black-Scholes risk neutral arguments for the valuation of options on assets that do not allow dynamic hedging, 4) There are fundamental informational limits preventing the convergence of the stochastic integral.[1]

There have been a couple of predecessors to the present thesis that Put-Call parity is sufficient constraint to enforce risk-neutrality, such as Derman and Taleb (2005), Haug and Taleb (2010), which were based on heuristic methods, robust though deemed hand-waving (Ruffino and Treussard, 2006). This paper uses a completely distribution-free, expectation-based approach and proves the risk-neutral argument without dynamic hedging, and without any distributional assumption, with solely two constraints: "horizontal", i.e. Put-Call Parity, and "vertical", i.e. the different valuations across strike prices deliver a probability measure (Dupire, 1994), which is shown to be unique. The only economic assumption made is that the forward is tradable by cash-and-carry style arbitrage — in the absence of such forward it is futile to discuss standard option pricing.

Aside from the cash and carry arbitrage, we make no assumption of market completeness. Options are not redundant securities and remain so.[2]

II. Proof

Define $C(S_{t_0}, K, t)$ and $P(S_{t_0}, K, t)$ as European-style call and put with strike price K, respectively, with expiration t, and S_0 as an underlying security at times t_0, $t \geq t_0$, and S_t the possible value of the underlying security at time t.

Define $r = \frac{1}{t-t_0}\int_{t_0}^{t} r_s \mathrm{d}s$, the return of a risk-free money market fund and $\delta = \frac{1}{t-t_0}\int_{t_0}^{t} \delta_s \mathrm{d}s$ the payout of the asset (continuous dividend for a stock, foreign interest for a currency).

We have the arbitrage forward price F_t^Q:

$$F_t^Q = S_0 \frac{(1+r)^{(t-t_0)}}{(1+\delta)^{(t-t_0)}} \approx S_0\, e^{(r-\delta)^{(t-t_0)}} \tag{1}$$

by arbitrage, see Keynes 1924. We thus call F_t^Q the future (or forward) price obtained by arbitrage, at the risk-neutral rate. Let F_t^P be the future requiring a risk-associated "expected return" m, with expected forward price:

$$F_t^P = S_0(1+m)^{(t-t_0)} \approx S_0\, e^{m\,(t-t_0)} \tag{2}$$

Remark: *By arbitrage, all tradable values of the forward price given S_{t_0} need to be equal to F_t^Q.*

"Tradable" here does not mean "traded", only subject to arbitrage replication by "cash and carry", that is, borrowing cash and owning the secutity yielding d if the embedded forward return diverges from r.

Define $\Omega = [0, \infty) = A_K \cup A_K^c$ where $A_K = [0, K)$ and $A_K^c = [K, \infty)$.

Consider a class of standard (simplified) probability spaces (Ω, μ_i) indexed by i, where μ_i is a probability measure, i.e., satisfying $\int_\Omega \mathrm{d}\mu_i = 1$.

Theorem 1. *For a given maturity T, there is a unique measure μ_Q that prices European puts and calls by expectation of*

[1]Further, in a case of scientific puzzle, the exact formula called "Black-Scholes-Merton" was written down (and used) by Edward Thorp in a heuristic derivation by expectation that did not require dynamic hedging.

[2]The famed Hakkanson paradox is as follows: if markets are complete and options are redudant, why would someone need them? If markets are incomplete, we may need options but how can we price them? This discussion may have provided a solution to the paradox: markets are incomplete *and* we can price options.

Table I
COMPARISON

	Black-Scholes Merton	**Put-Call Parity**
Type	Continuous rebalancing.	Interpolative static hedge.
Market Assumptions	1) Continuous Markets, no gaps, no jumps. 2) Ability to borrow and lend underlying asset for all dates. 3) No transaction costs in trading asset.	1) Gaps and jumps acceptable. Continuous Strikes, or acceptable number of strikes. 2) Ability to borrow and lend underlying asset for single forward date. 3) Low transaction costs in trading options.
Probability Distribution	Requires all moments to be finite. Excludes slowly varying distributions	Requires finite 1^{st} moment (infinite variance is acceptable).
Market Completeness	Achieved through dynamic completeness	Not required (in the traditional sense)
Realism of Assumptions	Low	High
Convergence	In probability (uncertain; one large jump changes expectation)	Pointwise
Fitness to Reality	Only used after "fudging" standard deviations per strike.	Portmanteau, adapted to reality

terminal payoff. This measure is risk-neutral in the sense that it prices the forward F_t^Q.

Lemma 1. *For a given maturity T, there exist two measures μ_1 and μ_2 for European calls and puts of the same maturity and same underlying security associated with the valuation by expectation of terminal payoff, which are unique such that, for any call and put of strike K, we have:*

$$C = \int_\Omega f_C \, \mathrm{d}\mu_1 \,, \tag{3}$$

and

$$P = \int_\Omega f_P \, \mathrm{d}\mu_2 \,, \tag{4}$$

respectively, and where f_C and f_P are $(S_t - K)^+$ and $(K - S_t)^+$ respectively.

Proof. For clarity, set r and δ to 0 without a loss of generality. By Put-Call Parity Arbitrage, a positive holding of a call ("long") and negative one of a put ("short") replicates a tradable forward; because of P/L variations, using positive sign for long and negative sign for short:

$$C(S_{t_0}, K, t) - P(S_{t_0}, K, t) + K = F_t^P \tag{5}$$

necessarily since F_t^P is tradable.

Put-Call Parity holds for all strikes, so:

$$C(S_{t_0}, K+\Delta K, t) - P(S_{t_0}, K+\Delta K, t) + K + \Delta K = F_t^P \tag{6}$$

for all $K \in \Omega$

Now a Call spread in quantities $\frac{1}{\Delta K}$, expressed as

$$C(S_{t_0}, K, t) - C(S_{t_0}, K + \Delta K, t),$$

delivers \$1 if $S_t > K + \Delta K$, 0 if $S_t < K$, and the quantity times $S_t - K$ if $K \leq S_t \leq K + \Delta K$, that is between 0 and \$1. Likewise, consider the converse argument for a put, with $\Delta K < S_t$.

At the limit, for $\Delta K \to 0$

$$\frac{\partial C(S_{t_0}, K, t)}{\partial K} = -\int_{A_K^c} \mathrm{d}\mu_1 \tag{7}$$

by the same argument:

$$\frac{\partial P(S_{t_0}, K, t)}{\partial K} = \int_{A_K} \mathrm{d}\mu_2 = 1 - \int_{A_K^c} \mathrm{d}\mu_2 \tag{8}$$

As semi-intervals generate the whole Borel σ-algebra on Ω, this shows that $\mu_1$and μ_2 are unique.

□

Lemma 2. *The probability measures of puts and calls are the same, namely for each Borel set A in Ω, $\mu_1(A) = \mu_2(A)$.*

Proof. Combining Equations 5 and 6, dividing by $\frac{1}{\Delta K}$ and taking $\Delta K \to 0$:

$$-\frac{\partial C(S_{t_0}, K, t)}{\partial K} + \frac{\partial P(S_{t_0}, K, t)}{\partial K} = 1 \tag{9}$$

for all values of K, so

$$\int_{A_K^c} \mathrm{d}\mu_1 = \int_{A_K^c} \mathrm{d}\mu_2 \quad (10)$$

hence $\mu_1(A_K) = \mu_2(A_K)$ for all $K \in [0, \infty)$. This equality being true for any semi-interval, it extends to any Borel set.

□

Lemma 3. *Puts and calls are required, by static arbitrage, to be evaluated at same as risk-neutral measure μ_Q as the tradable forward.*

Proof.

$$F_t^P = \int_\Omega F_t \, \mathrm{d}\mu_Q \quad (11)$$

From Equation 5

$$\int_\Omega f_C(K) \, \mathrm{d}\mu_1 - \int_\Omega f_P(K) \, \mathrm{d}\mu_1 = \int_\Omega F_t \, \mathrm{d}\mu_Q - K \quad (12)$$

Taking derivatives on both sides, and since $f_C - f_P = S_0 + K$, we get the Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mu_Q}{\mathrm{d}\mu_1} = 1 \quad (13)$$

for all values of K.

□

III. COMMENT

We have replaced the complexity and intractability of dynamic hedging with a simple, more benign interpolation problem, and explained the performance of pre-Black-Scholes option operators using simple heuristics and rules.

Options can remain non-redundant and markets incomplete: we are just arguing here for risk-neutral pricing (at the level of the expectation of the probability measure), nothing more. But this is sufficient for us to use any probability distribution with finite first moment, which includes the Lognormal, which recovers Black Scholes.

A final comparison. In dynamic heding, missing a single hedge, or encountering a single gap (a tail event) can be disastrous —as we mentioned, it requires a series of assumptions beyond the mathematical, in addition to severe and highly unrealistic constraints on the mathematical. Under the class of fat tailed distributions, increasing the frequency of the hedges does not guarantee reduction of risk. Further, the standard dynamic hedging argument requires the exact specification of the *risk-neutral* stochastic process between t_0 and t, something econometrically unwieldy, and which is generally reverse engineered from the price of options, as an arbitrage-oriented interpolation tool rather than as a representation of the process.

Here, in our Put-Call Parity based methodology, our ability to track the risk neutral distribution is guaranteed by adding strike prices, and since probabilities add up to 1, the degrees of freedom that the recovered measure μ_Q has in the gap area between a strike price K and the next strike up, $K + \Delta K$, are severely reduced, since the measure in the interval is constrained by the difference $\int_{A_K}^c \mathrm{d}\mu - \int_{A_{K+\Delta K}}^c \mathrm{d}\mu$. In other words, no single gap between strikes can significantly affect the probability measure, even less the first moment , which is the exact opposite of dynamic hedging. In fact it is no different from standard kernel smoothing methods for statistical samples, but applied to the distribution across strikes.[3]

The assumption about the presence of strike prices constitutes a natural condition: conditional on having a *practical* discussion about options, options strikes need to exist. Further, as it is the experience of the author, market-makers can add over-the-counter strikes at will, should they need to do so.

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[3]For methods of interpolation of implied probability distribution between strikes, see Avellaneda et al.(1997).